Path Integrals in Quantum Field Theory

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Abstract

We discuss the path integral formulation of quantum mechanics and use it to derive the S matrix in terms of Feynman diagrams. We generalize to quantum field theory, and derive the generating functional $Z[J]$ and n-point correlation functions for free scalar field theory. We develop the generating functional for self-interacting fields and discuss ϕ^4 and ϕ^3 theory.

1 Introduction

> Thirty-one years ago, Dick Feynman told me about his 'sum over histories' version of quantum mechanics. 'The electron does anything it likes', he said. 'It goes in any direction at any speed, forward and backward in time, however it likes, and then you add up the amplitudes and it gives you the wavefunction.' I said to him, 'You're crazy'. But he wasn't.
>
> F.J. Dyson[1]

When we write down Feynman diagrams in quantum field theory, we proceed with the mind-set that our system will take on every configuration imaginable in traveling from the initial to final state. Photons will split in to electrons that recombine into different photons, leptons and anti-leptons will annihilate one another and the resulting energy will be used to create leptons of a different flavour; anything that can happen, will happen. Each distinct history can be thought of as a path through the configuration space that describes the state of the system at any given time. For quantum field theory, the configuration space is a Fock space where each vector represents the number of each type of particle with momentum $\mathbf{k}$. The key to the whole thing, though, is that each path that the system takes comes with a probabilistic *amplitude*. The probability that a system in some initial state will end up in some final state is given as a sum over the amplitudes associated with each path connecting the initial and final positions in the Fock space. Hence the perturbative expansion of scattering amplitudes in terms of Feynman diagrams, which represent all the possible ways the system can behave.

But quantum field theory is rooted in ordinary quantum mechanics; the essential difference is just the number of degrees of freedom. So what is the analogue of this "sum over histories" in ordinary quantum mechanics? The answer comes from the path integral formulation of quantum mechanics, where the amplitude that a particle at a given point in ordinary space will be found at some other point in the future is a sum over the amplitudes associated with all possible trajectories joining the initial and final positions. The amplitude associated with any given path is just e^{iS}, where S is the classical action $S = \int L(q, \dot{q})\, dt$. We will derive this result from the canonical formulation of quantum mechanics, using, for example, the time-dependent Schrödinger equation. However, if one *defines* the amplitude associated with a given trajectory as e^{iS}, then it is possible to derive the Schrödinger equation[2]. We can even "derive" the classical principle of least action from the quantum amplitude e^{iS}. In other words, one can view the amplitude of traveling from one point to another, usually called the propagator, as the fundamental object in quantum theory, from which the wavefunction follows. However, this formalism is of little

[1] Shamelessly lifted from page 154 of Ryder [1].

[2] Although, the procedure is only valid for velocity-independent potentials, see below.

use in quantum mechanics because state-vector methods are so straightforward; the path integral formulation is a little like using a sledge-hammer to kill a fly.

However, the situation is a lot different when we consider field theory. The generalization of path integrals leads to a powerful formalism for calculating various observables of quantum fields. In particular, the idea that the propagator Z is the central object in the theory is fleshed out when we discover that all of the n-point functions of an interacting field theory can be derived by taking derivatives of Z. This gives us an easy way of calculating scattering amplitudes that has a natural interpretation in terms of Feynman diagrams. All of this comes without assuming commutation relations, field decompositions or anything else associated with the canonical formulation of field theory. Our goal in this paper will to give an account of how path integrals arise in ordinary quantum mechanics and then generalize these results to quantum field theory and show how one can derive the Feynman diagram formalism in a manner independent of the canonical formalism.

2 Path integrals in quantum mechanics

To motivate our use of the path integral formalism in quantum field theory, we demonstrate how path integrals arise in ordinary quantum mechanics. Our work is based on section 5.1 of Ryder [1] and chapter 3 of Baym [2]. We consider a quantum system represented by the Heisenberg state vector $|\psi\rangle$ with one coordinate degree of freedom q and its conjugate momentum p. We adopt the notation that the Schrödinger representation of any given state vector $|\phi\rangle$ is given by

$$|\phi, t\rangle = e^{-iHt}|\phi\rangle, \tag{1}$$

where $H = H(q, p)$ is the system Hamiltonian. According to the probability interpretation of quantum mechanics, the wavefunction $\psi(q, t)$ is the projection of $|\psi, t\rangle$ onto an eigenstate of position $|q\rangle$. Hence

$$\psi(q, t) = \langle q|\psi, t\rangle = \langle q, t|\psi\rangle, \tag{2}$$

where we have defined

$$|q, t\rangle = e^{iHt}|q\rangle. \tag{3}$$

$|q\rangle$ satisfies the completeness relation

$$\langle q|q'\rangle = \delta(q - q'), \tag{4}$$

which implies

$$\langle q|\psi\rangle = \int dq' \langle q|q'\rangle\langle q'|\psi\rangle, \tag{5}$$

or

$$1 = \int dq' |q'\rangle\langle q'|. \tag{6}$$

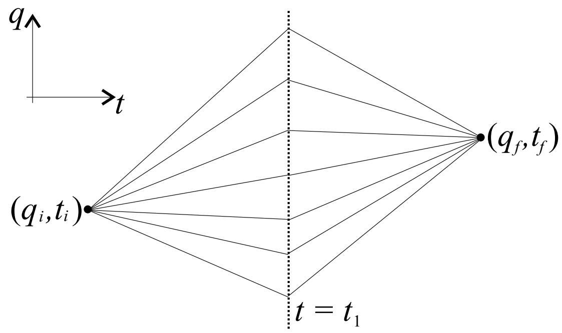

Figure 1: The various two-legged paths that are considered in the calculation of $\langle q_f, t_f | q_i, t_i \rangle$

Multiplying by $e^{iHt'}$ on the left and $e^{-iHt'}$ on the right yields that

$$1 = \int dq' |q', t'\rangle\langle q', t'|. \tag{7}$$

Now, using the completeness of the $|q, t\rangle$ basis, we may write

$$\begin{aligned}\psi(q_f, t_f) &= \int dq_i \langle q_f, t_f | q_i, t_i \rangle \langle q_i, t_i | \psi \rangle \\ &= \int dq_i \langle q_f, t_f | q_i, t_i \rangle \psi(q_i, t_i). \end{aligned} \tag{8}$$

The quantity $\langle q_f, t_f | q_i, t_i \rangle$ is called the *propagator* and it represents the probability amplitudes (expansion coefficients) associated with the decomposition of $\psi(q_f, t_f)$ in terms of $\psi(q_i, t_i)$. If $\psi(q_i, t_i)$ has the form of a spatial delta function $\delta(q_0)$, then $\psi(q_f, t_f) = \langle q_f, t_f | q_0, t_i \rangle$. That is, if we know that the particle is at q_0 at some time t_i, then the probability that it will be later found at a position q_f at a time t_f is

$$P(q_f, t_f; q_0, t_i) = |\langle q_f, t_f | q_i, t_0 \rangle|^2. \tag{9}$$

It is for this reason that we sometimes call the propagator a *correlation function*.

Now, using completeness, it is easily seen that the propagator obeys a *composition equation*:

$$\langle q_f, t_f | q_i, t_i \rangle = \int dq_1 \langle q_f, t_f | q_1, t_1 \rangle \langle q_1, t_1 | q_i, t_i \rangle. \tag{10}$$

This can be understood by saying that the probability amplitude that the position of the particle is q_i at time t_i and q_f at time t_f is equal to the sum over q_1 of the probability that the particle traveled from q_i to q_1 (at time t_1) and then on to q_f. In other words, the probability amplitude that a particle initially at q_i will later be seen at q_f is the sum of the probability amplitudes associated with all possible

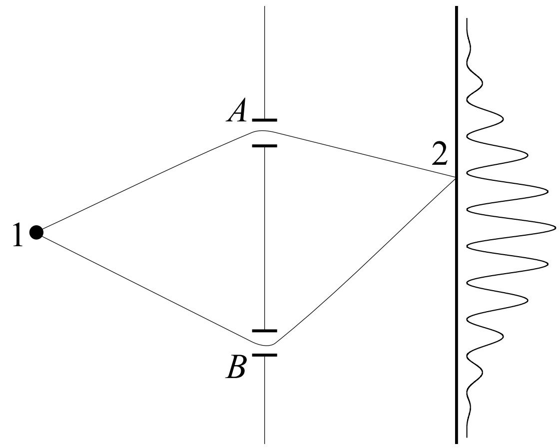

Figure 2: The famous double-slit experiment

two-legged paths between q_i and q_f, as seen in figure 1. This is the meaning of the oft-quoted phrase: "motion in quantum mechanics is considered to be a sum over paths". A particularly neat application comes from the double slit experiment that introductory texts use to demonstrate the wave nature of elementary particles. The situation is sketched in figure 2. We label the initial point (q_i, t_i) as 1 and the final point (q_f, t_f) as 2. The amplitude that the particle (say, an electron) will be found at 2 is the sum of the amplitude of the particle traveling from 1 to A and then to 2 and the amplitude of the particle traveling from 1 to B and then to 2. Mathematically, we say that

$$\langle 2|1\rangle = \langle 2|A\rangle\langle A|1\rangle + \langle 2|B\rangle\langle B|1\rangle. \tag{11}$$

The presence of the double-slit ensures that the integral in (10) reduces to the two-part sum in (11). When the probability $|\langle 2|1\rangle|^2$ is calculated, interference between the $\langle 2|A\rangle\langle A|1\rangle$ and $\langle 2|B\rangle\langle B|1\rangle$ terms will create the classic intensity pattern on the screen.

There is no reason to stop at two-legged paths. We can just as easily separate the time between t_i and t_f into n equal segments of duration $\tau = (t_f - t_i)/n$. It then makes sense to relabel $t_0 = t_i$ and $t_n = t_f$. The propagator can be written as

$$\langle q_n, t_n|q_0, t_0\rangle = \int dq_1 \cdots dq_{n-1}\, \langle q_n, t_n|q_{n-1}, t_{n-1}\rangle \cdots \langle q_1, t_1|q_0, t_0\rangle. \tag{12}$$

We take the limit $n \to \infty$ to obtain an expression for the propagator as a sum over infinite-legged paths, as seen in figure 3. We can calculate the propagator for small time intervals $\tau = t_{j+1} - t_j$ for some j between 1 and $n-1$. We have

$$\langle q_{j+1}, t_{j+1}|q_j, t_j\rangle = \langle q_{j+1}|e^{-iHt_{j+1}}e^{+iHt_j}|q_j\rangle$$

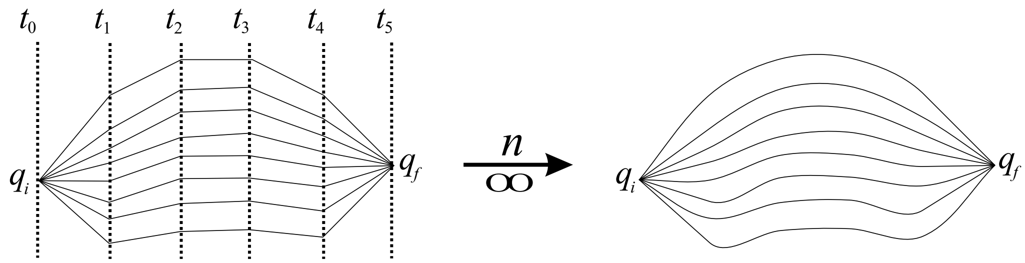

Figure 3: The continuous limit of a collection of paths with a finite number of legs

$$
\begin{aligned}
&= \langle q_{j+1}|(1 - iH\tau + O(\tau^2)|q_j\rangle \\
&= \delta(q_{j+1} - q_j) - i\tau\langle q_{j+1}|H|q_j\rangle \\
&= \frac{1}{2\pi}\int dp\, e^{ip(q_{j+1}-q_j)} - \frac{i\tau}{2m}\langle q_{j+1}|p^2|q_j\rangle \\
&\quad -i\tau\langle q_{j+1}|V(q)|q_j\rangle,
\end{aligned}
\tag{13}
$$

where we have assumed a Hamiltonian of the form

$$
H(p,q) = \frac{p^2}{2m} + V(q). \tag{14}
$$

Now,

$$
\langle q_{j+1}|p^2|q_j\rangle = \int dp\, dp'\, \langle q_{j+1}|p'\rangle\langle p'|p^2|p\rangle\langle p|q_j\rangle, \tag{15}
$$

where $|p\rangle$ is an eigenstate of momentum such that

$$
p|p\rangle = |p\rangle p, \quad \langle q|p\rangle = \frac{1}{\sqrt{2\pi}}e^{ipq}, \quad \langle p|p'\rangle = \delta(p-p'). \tag{16}
$$

Putting these expressions into (15) we get

$$
\langle q_{j+1}|p^2|q_j\rangle = \frac{1}{2\pi}\int dp\, p^2 e^{ip(q_{j+1}-q_j)}, \tag{17}
$$

where we should point out that p^2 is a number, not an operator. Working on the other matrix element in (13), we get

$$
\begin{aligned}
\langle q_{j+1}|V(q)|q_j\rangle &= \langle q_{j+1}|q_j\rangle V(q_j) \\
&= \delta(q_{j+1} - q_j)V(q_j) \\
&= \frac{1}{2\pi}\int dp\, e^{ip(q_{j+1}-q_j)}V(q_j).
\end{aligned}
$$

Putting it all together

$$\begin{aligned}\langle q_{j+1}, t_{j+1} | q_j, t_j \rangle &= \frac{1}{2\pi} \int dp\, e^{ip(q_{j+1}-q_j)} \left[1 - i\tau H(p, q_j) + O(\tau^2)\right] \\ &= \frac{1}{2\pi} \int dp \exp\left[i\tau \left(p\, \frac{\Delta q_j}{\tau} - H(p, q_j)\right)\right],\end{aligned}$$

where $\Delta q_j \equiv q_{j+1} - q_j$. Substituting this expression into (12) we get

$$\langle q_n, t_n | q_0, t_0 \rangle = \int dp_0 \prod_{i=1}^{n-1} \frac{dq_i\, dp_i}{2\pi} \exp\left[i \sum_{j=0}^{n-1} \tau \left(p_j\, \frac{\Delta q_j}{\tau} - H(p_j, q_j)\right)\right]. \tag{18}$$

In the limit $n \to \infty$, $\tau \to 0$, we have

$$\sum_{j=0}^{n-1} \tau \to \int_{t_0}^{t_n} dt, \quad \frac{\Delta q_j}{\tau} \to \frac{dq}{dt} = \dot{q}, \quad dp_0 \prod_{i=1}^{n} \frac{dq_i\, dp_i}{2\pi} \to [dq]\, [dp], \tag{19}$$

and

$$\langle q_n, t_n | q_0, t_0 \rangle = \int [dq]\, [dp] \exp\left\{i \int_{t_0}^{t_n} dt\, [p\, \dot{q} - H(p, q)]\right\}. \tag{20}$$

The notation $[dq]\, [dp]$ is used to remind us that we are integrating over *all possible paths* $q(t)$ and $p(t)$ that connect the points (q_0, t_0) and (q_n, t_n). Hence, we have succeed in writing the propagator $\langle q_n, t_n | q_0, t_0 \rangle$ as a *functional integral* over the all the phase space trajectories that the particle can take to get from the initial to the final points. It is at this point that we fully expect the reader to scratch their heads and ask: what exactly is a functional integral? The simple answer is a quantity that arises as a result of the limiting process we have already described. The more complicated answer is that functional integrals are beasts of a rather vague mathematical nature, and the arguments as to their standing as well-behaved entities are rather nebulous. The philosophy adopted here is in the spirit of many mathematically controversial manipulations found in theoretical physics: we assume that everything works out alright.

The argument of the exponential in (20) ought to look familiar. We can bring this out by noting that

$$\begin{aligned}\frac{1}{2\pi} \int dp_i\, e^{i\tau\left[p_i \frac{\Delta q_i}{\tau} - H(p_i, q_i)\right]} &= \frac{1}{2\pi} \exp\left\{i\tau \left[\frac{m}{2} \left(\frac{\Delta q_i}{\tau}\right)^2 - V(q_i)\right]\right\} \\ &\quad \times \int dp_i \exp\left[-\frac{i\tau}{2m} \left(p - \frac{m\Delta q_i}{\tau}\right)^2\right] \\ &= \left(\frac{m}{2\pi i\tau}\right)^{1/2} \exp\left\{i\tau \left[\frac{m}{2} \left(\frac{\Delta q_i}{\tau}\right)^2 - V(q_i)\right]\right\}.\end{aligned}$$

Using this result in (18) we obtain

$$\begin{aligned}\langle q_n, t_n | q_0, t_0 \rangle \;=\; & \left(\frac{m}{2\pi i \tau}\right)^{n/2} \int \prod_{i=1}^{n-1} dq_i \exp\left\{ i \sum_{j=0}^{n-1} \tau \left[\frac{m}{2} \left(\frac{\Delta q_j}{\tau} \right)^2 - V(q_j) \right] \right\} \\ & \to N \int [dq] \exp\left[i \int_{t_0}^{t_n} dt \left(\frac{1}{2} m \dot{q}^2 - V(q) \right) \right], \end{aligned} \tag{21}$$

where the limit is taken, as usual, for $n \to \infty$ and $\tau \to 0$. Here, N is an infinite constant given by

$$N = \lim_{n \to \infty} \left(\frac{m}{2\pi i \tau} \right)^{n/2}. \tag{22}$$

We won't worry too much about the fact that N diverges because we will later normalize our transition amplitudes to be finite. Recognizing the Lagrangian $L = T - V$ in equation (21), we have

$$\langle q_n, t_n | q_0, t_0 \rangle = N \int [dq] \exp\left[i \int_{t_0}^{t_n} L(q, \dot{q})\, dt \right] = N \int [dq]\, e^{iS[q]}, \tag{23}$$

where S is the classical action, given as a *functional* of the trajectory $q = q(t)$. Hence, we see that the propagator is the sum over paths of the amplitude $e^{iS[q]}$, which is the amplitude that the particle follows a given trajectory $q(t)$. Historically, Feynman demonstrated that the Schrödinger equation could be derived from equation (23) and tended to regard the relation as the fundamental quantity in quantum mechanics. However, we have assumed in our derivation that the potential is a function of q and not p. If we do indeed have velocity-dependent potentials, (23) fails to recover the Schrödinger equation. We will not go into the details of how to fix the expression here, we will rather heuristically adopt the generalization of (23) for our later work in with quantum fields[3].

An interesting consequence of (23) is seen when we restore $\hbar$. Then

$$\langle q_n, t_n | q_0, t_0 \rangle = N \int [dq]\, e^{iS[q]/\hbar}. \tag{24}$$

The classical limit is obtained by taking $\hbar \to 0$. Now, consider some trajectory $q_0(t)$ and neighbouring trajectory $q_0(t) + \delta q(t)$, as shown in figure 4. The action evaluated along q_0 is S_0 while the action along $q_0 + \delta q$ is $S_0 + \delta S$. The two paths will then make contributions $\exp(iS_0/\hbar)$ and $\exp[i(S_0 + \delta S)/\hbar]$ to the propagator. For $\hbar \to 0$, the phases of the exponentials will become completely disjoint and the contributions will in general destructively interfere. That is, unless $\delta S = 0$ in which case all neighbouring paths will constructively interfere. Therefore, in the classical limit the propagator will be non-zero for points that may be connected by a trajectory

[3]The generalization of velocity-dependent potentials to field theory involves the quantization of non-Abelian gauge fields

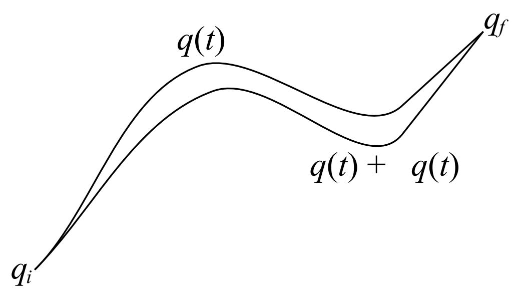

Figure 4: Neighbouring particle trajectories. If the action evaluated along $q(t)$ is stationary (i.e. $\delta S = 0$), then the contribution of $q(t)$ and it's neighbouring paths $q(t) + \delta q(t)$ to the propagator will constructively interfere and reconstruct the classical trajectory in the limit $\hbar \to 0$

.

satisfying $\delta S[q]|_{q=q_0}$; i.e. for paths connected by *classical* trajectories determined by Newton's 2^{nd} law. We have hence seen how the classical principle of least action can be understood in terms of the path integral formulation of quantum mechanics and a corresponding principle of *stationary phase*.

3 Perturbation theory, the scattering matrix and Feynman rules

In practical calculations, it is often impossible to solve the Schrödinger equation exactly. In a similar manner, it is often impossible to write down analytic expressions for the propagator $\langle q_f, t_f | q_i, t_i \rangle$ for general potentials $V(q)$. However, if one assumes that the potential is small and that the particle is nearly free, one makes good headway by using perturbation theory. We follow section 5.2 in Ryder [1].

In this section, we will go over from the general configuration coordinate q to the more familiar x, which is just the position of the particle in a one-dimensional space. The extension to higher dimensions, while not exactly trivial, is not difficult to do. We assume that the potential that appears in (23) is "small", so we may perform an expansion

$$\exp\left[-i\int_{t_0}^{t_n} V(x,t)\,dt\right] = 1 - i\int_{t_0}^{t_n} V(x,t)\,dt - \frac{1}{2!}\left[\int_{t_0}^{t_n} V(x,t)\,dt\right]^2 + \cdots. \quad (25)$$

We adopt the notation that $K = K(x_n, t_n; x_0, t_0) = \langle x_n, t_n | x_0, t_0 \rangle$. Inserting the expansion (25) into the propagator, we see that K possesses and expansion of the form:

$$K = K_0 + K_1 + K_2 + \cdots \quad (26)$$

The K_0 term is

$$K_0 = N \int [dx] \, \exp\left[i \int \frac{1}{2} m\dot{x}^2 \, dt\right]. \qquad (27)$$

If we turned off the potential, the full propagator would reduce to K_0. It is for this reason that we call K_0 the *free particle propagator*, it represents the amplitude that a free particle known to be at x_0 at time t_0 will later be found at x_n at time t_n. Going back to the discrete expression:

$$K_0 = \lim_{n\to\infty} \left(\frac{m}{2\pi i\tau}\right)^{n/2} \int \prod_{i=1}^{n-1} dx_i \, \exp\left[\frac{im\tau}{2} \sum_{j=0}^{n-1} (x_{j+1} - x_j)^2\right]. \qquad (28)$$

This is a doable integral because the argument of the exponential is a simple quadratic form. We can hence diagonalize it by choosing an appropriate rotation of the x_j Cartesian variables of integration. Conversely, we can start calculating for $n = 2$ and solve the general n case using induction. The result is

$$K_0 = \lim_{n\to\infty} \left(\frac{m}{2\pi i\tau}\right)^{n/2} \frac{1}{n^{1/2}} \left(\frac{2\pi i\tau}{m}\right)^{(n-1)/2} \exp\left[\frac{im(x_n - x_0)^2}{2n\tau}\right]. \qquad (29)$$

Now, $(t_n - t_0)/n = \tau$, so we finally have

$$K_0(x_n - x_0, t_n - t_0) = \left[\frac{m}{2\pi i(t_n - t_0)}\right]^{1/2} \exp\left[\frac{im(x_n - x_0)^2}{2(t_n - t_0)}\right], \quad t_n > t_0. \qquad (30)$$

Here, we've noted that the substitution $n\tau = (t_n - t_0)$ is only valid for $t_n > t_0$. In fact, if K_0 is non-zero for $t_n > t_0$ it must be zero for $t_0 > t_n$. To see this, we note that the calculation of K_0 involved integrations of the form:

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{i\alpha x^2} \, dx &= \frac{1}{2} \int_0^{\infty} e^{i\alpha x^2} \, dx \\
&= \frac{i^{-1/2}}{4} \int_0^{i\infty} \frac{e^{\alpha s}}{s^{1/2}} \, ds \\
&= \frac{i^{-1/2}}{4} \int_{-i\infty}^{i\infty} \Theta(-is) \frac{e^{\alpha s}}{s^{1/2}} \, ds,
\end{aligned}$$

where $\alpha \propto \text{sign}(\tau) = \text{sign}(t_n - t_0)$. Now, we can either choose the branch of $s^{-1/2}$ to be in either the left- or righthand part of the complex s-plane. But, we need to complete the contour in the lefthand plane if $\alpha > 0$ and the righthand plane if $\alpha < 0$. Hence, the integral can only be non-zero for one case of the sign of α. The choice we have implicitly made is the the integral is non-zero for $\alpha \propto (t_n - t_0) > 0$, hence it must vanish for $t_n < t_0$. When we look at equation (8) we see that K_0 is little more than a type of kernel for the integral solution of the free-particle Schrödinger equation, which is really a statement about Huygen's principle. Our choice of K_0 obeys causality in that the configuration of the field at prior times determines the form of the field in the present. We have hence found a retarded propagator. The other choice for the boundary conditions obeyed by K_0 yields the advanced propagator and a version of Huygen's principle where future field configurations determine the present state. The moral of the story is that, if we choose a propagator that obeys casuality, we are justified in writing

$$K_0(x,t) = \Theta(t)\left[\frac{m}{2\pi i t}\right]^{1/2}\exp\left[\frac{imx^2}{2t}\right]. \tag{31}$$

Now, we turn to the calculation of K_1:

$$K_1 = -iN\int [dx]\,\exp\left[i\int \frac{1}{2}m\dot{x}^2\,dt\right]\int dt\,V(x(t),t). \tag{32}$$

Moving again to the discrete case:

$$K_1 = -i\beta^{n/2}\int dx_1\cdots dx_{n-1}\,\exp\left[\frac{im\tau}{2}\sum_{j=0}^{n-1}(x_{j+1}-x_j)^2\right]\sum_{i=1}^{n-1}\tau V(x_i,t_i), \tag{33}$$

where $\beta = m/2\pi i\tau$ and the limit $n\to\infty$ is understood. Let's take the sum over i (which has replaced the integral over t) in front of the spatial integrals. Also, let's split up the sum over j in the exponential to a sum running from 0 to $i-1$ and a sum running from i to $n-1$. Then

$$\begin{aligned} K_1 = & -i\sum_{i=1}^{n-1}\tau\int dx_i\,\beta^{i/2}\int dx_1\cdots dx_{i-1}\exp\left[\frac{im\tau}{2}\sum_{j=0}^{i-1}(x_{j+1}-x_j)^2\right]V(x_i,t_i) \\ & \times\beta^{(n-i)/2}\int dx_{i+1}\cdots dx_{n-1}\exp\left[\frac{im\tau}{2}\sum_{j=i}^{n-1}(x_{j+1}-x_j)^2\right]. \end{aligned} \tag{34}$$

We recognize two factors of the free-particle propagator in this expression, which allows us to write

$$K_1 = -i\sum_{i=1}^{n-1}\tau\int dx\,K_0(x-x_0,t_i-t_0)V(x,t_i)K_0(x_n-x,t_n-t_i). \tag{35}$$

Now, we can replace $\sum_{i=1}^{n-1}\tau$ by $\int_{t_0}^{t_n}dt$ and $t_i\to t$ in the limit $n\to\infty$. Since $K_0(x-x_0,t-t_0)=0$ for $t<t_0$ and $K_0(x_n-x,t_n-t)$ for $t>t_n$, we can extend the limits on the time integration to $\pm\infty$. Hence,

$$K_1 = -i\int dx\,dt\,K_0(x_n-x,t_n-t)V(x,t)K_0(x-x_0,t-t_0). \tag{36}$$

In a similar fashion, we can derive the expression for K_2:

$$K_2 = \frac{(-i)^2}{2!}\beta^{n/2}\int dx_1\cdots dx_{n-1}\,\exp\left[\frac{im\tau}{2}\sum_{j=0}^{n-1}(x_{j+1}-x_j)^2\right] \tag{37}$$

$$\times\sum_{i=1}^{n-1}\tau V(x_i,t_i)\sum_{k=1}^{n-1}\tau V(x_k,t_k). \tag{38}$$

We would like to play the same trick that we did before by splitting the sum over j into three parts with the potential terms sandwiched in between. We need to construct the middle j sum to go from an early time to a late time in order to replace it with a free-particle propagator. But the problem is, we don't know whether t_i comes before or after t_k. To remedy this, we split the sum over k into a sum from 1 to $i-1$ and then a sum from i to $n-1$. In each of those sums, we can easily determine which comes first: t_i or t_k. Going back to the continuum limit:

$$\begin{aligned}K_2 = \frac{(-i)^2}{2!}\int dx_1\,dx_2\int_{t_0}^{t_n}dt_1\Bigg[\int_{t_0}^{t_1}&dt_2K_0(x_n-x_1,t_n-t_1)\\ &\times V(x_1,t_1)K_0(x_1-x_2,t_1-t_2)V(x_2,t_2)K_0(x_2-x_0,t_2-t_0)\\ &+\int_{t_1}^{t_n}dt_2K_0(x_n-x_2,t_n-t_2)V(x_2,t_2)K_0(x_2-x_1,t_2-t_1)\\ &\times V(x_1,t_1)K_0(x_1-x_0,t_1-t_0)\Bigg]\end{aligned} \tag{39}$$

But, we can extend the limits on the t_2 integration to $t_0 \to t_n$ by noting the middle propagator is zero for $t_2 > t_1$. Similarly, the t_2 limits on the second integral can be extended by observing the middle propagator vanishes for $t_1 > t_2$. Hence, both integrals are the same, which cancels the $1/2!$ factor. Using similar arguments, the limits of both of the remaining time integrals can be extended to $\pm\infty$ yielding our final result:

$$\begin{aligned}K_2 = (-i)^2\int &dx_1\,dx_2\,dt_1\,dt_2\,K_0(x_n-x_2,t_n-t_2)V(x_2,t_2)\\ &\times K_0(x_2-x_1,t_2-t_1)V(x_1,t_1)K_0(x_1-x_0,t_1-t_0).\end{aligned} \tag{40}$$

Higher order contributions to the propagator follow in a similar fashion. The general j^{th} order correction to the free propagator is

$$\begin{aligned}K_j = (-i)^j\int &dx_1\ldots dx_j\,dt_1\ldots dt_j\,K_0(x_n-x_j,t_n-t_j)\\ &\times V(x_j)\cdots V(x_1)K_0(x_1-x_0,t_1-t_0).\end{aligned} \tag{41}$$

We would like to apply this formalism to scattering problems where we assume that the particle is initially in a plane wave state incident on some localized potential.

As $t \to \pm\infty$, we assume the potential goes to zero, which models the fact that the particle is far away from the scattering region in the distant past and the distant future. We go over from one to three dimensions and write

$$\begin{aligned}\psi(\mathbf{x}_f, t_f) = &\int d\mathbf{x}_i K_0(\mathbf{x}_f - \mathbf{x}_i, t_f - t_i)\psi(\mathbf{x}_i, t_i) \\ &-i\int d\mathbf{x}_i\, d\mathbf{x}\, dt\, K_0(\mathbf{x}_f - \mathbf{x}, t_f - t) \\ &\times V(\mathbf{x}, t) K_0(\mathbf{x} - \mathbf{x}_i, t - t_i)\psi(\mathbf{x}_i, t_i) + \cdots \end{aligned} \tag{42}$$

We push t_i into the distant past, where the effects of the potential may be ignored, and take the particle to be in a plane wave state:

$$\psi_{\rm in}(\mathbf{x}_i, t_i) = \frac{1}{\sqrt{V}} e^{-ip_i \cdot x_i}, \tag{43}$$

where we have used a box normalization with V being the volume of the box and $p_i \cdot x = E_i t_i - \mathbf{p}_i \cdot \mathbf{x}_i$. The "in" label on the wavefunction is meant to emphasize that it is the form of ψ before the particle moves into the scattering region. We want to calculate the first integral in (42) using the 3D generalization of (31):

$$K_0(\mathbf{x}, t) = -i\Theta(t)\left(\frac{\lambda}{\pi}\right)^{3/2} e^{\lambda \mathbf{x}^2}, \tag{44}$$

where $\lambda = im/2t$. Hence,

$$\begin{aligned}\int d\mathbf{x}_i K_0(\mathbf{x}_f - \mathbf{x}_i, t_f - t_i)\psi_{\rm in}(\mathbf{x}_i, t_i) = &-\frac{i}{\sqrt{V}}\left(\frac{\lambda}{\pi}\right)^{3/2} \\ &\times e^{-iE_i t_i}\int d\mathbf{x}_i e^{\lambda(\mathbf{x}_f - \mathbf{x}_i)^2 + i\mathbf{p}_i\cdot\mathbf{x}_i}. \end{aligned} \tag{45}$$

This integral reduces to $\psi_{\rm in}(\mathbf{x}_f, t_f)$ as should have been expected, because K_0 is the free particle propagator and must therefore propagate plane waves into the future without altering their form. We also push t_f into the infinite future where the effects of the potential can be ignored. Then,

$$\begin{aligned}\psi^+(\mathbf{x}_f, t_f) = &\,\psi_{\rm in}(\mathbf{x}_f, t_f) - i\int d\mathbf{x}_i\, d\mathbf{x}\, dt\, K_0(\mathbf{x}_f - \mathbf{x}, t_f - t) \\ &\times V(\mathbf{x}, t) K_0(\mathbf{x} - \mathbf{x}_i, t - t_i)\psi_{\rm in}(\mathbf{x}_i, t_i) + \cdots \end{aligned} \tag{46}$$

The "+" notation on ψ is there to remind us that ψ^+ is the form of the wave function after it interacts with the potential. What we really want to do is Fourier analyze $\psi^+(\mathbf{x}_f, t_f)$ into momentum eigenstates to determine the probability amplitude for a particle of momentum $\mathbf{p}_i$ becoming a particle of momentum $\mathbf{p}_f$ after interacting with the potential. Defining $\psi_{\rm out}(\mathbf{x}_f, t_f)$ as a state of momentum p_f in the distant future:

$$\psi_{\rm out}(\mathbf{x}_f, t_f) = \frac{1}{\sqrt{V}} e^{-ip_f \cdot x_f}, \tag{47}$$

we can write the amplitude for a transition from $\mathbf{p}_i$ to $\mathbf{p}_f$ as

$$S_{fi} = \langle \psi_{\rm out} | \psi^+ \rangle. \tag{48}$$

Inserting the unit operator $1 = \int d\mathbf{x}_f |\mathbf{x}_f, t_f\rangle\langle \mathbf{x}_f, t_f|$ into (48) and using the propagator expansion (46), we obtain

$$\begin{aligned} S_{fi} &= \delta(\mathbf{p}_f - \mathbf{p}_i) - i \int d\mathbf{x}_i \, d\mathbf{x}_f \, d\mathbf{x} \, dt \, \psi^*_{\rm out}(\mathbf{x}_f, t_f) K_0(\mathbf{x}_f - \mathbf{x}, t_f - t) \\ &\quad \times V(\mathbf{x}, t) K_0(\mathbf{x} - \mathbf{x}_i, t - t_i) \psi_{\rm in}(\mathbf{x}_i, t_i) + \cdots \end{aligned} \tag{49}$$

The amplitude S_{fi} is the fi component of what is known as the S or scattering matrix. This object plays a central rôle in scattering theory because it answers all the questions that one can experimentally ask about a physical scattering process. What we have done is expand these matrix elements in terms of powers of the scattering potential. Our expansion can be given in terms of *Feynman diagrams* according to the rules:

1. The vertex of this theory is attached to two legs and a spacetime point $(\mathbf{x}, t)$.
2. Each vertex comes with a factor of $-iV(\mathbf{x}, t)$.
3. The arrows on the lines between vertices point from the past to the future.
4. Each line going from $(\mathbf{x}, t)$ to $(\mathbf{x}', t')$ comes with a propagator $K_0(\mathbf{x}' - \mathbf{x}, t' - t)$.
5. The past external point comes with the wavefunction $\psi_{\rm in}(\mathbf{x}_i, t_i)$, the future one comes with $\psi^*_{\rm out}(\mathbf{x}_f, t_f)$.
6. All spatial coordinates and internal times are integrated over.

Using these rules, the S matrix element may be represented pictorially as in figure 5. We note that these rules are for configuration space only, but we could take Fourier transforms of all the relevant quantities to get momentum space rules. Obviously, the Feynman rules for the Schrödinger equation do not result in a significant simplification over the raw expression (49), but it is important to notice *how* they were derived: using simple and elegant path integral methods.

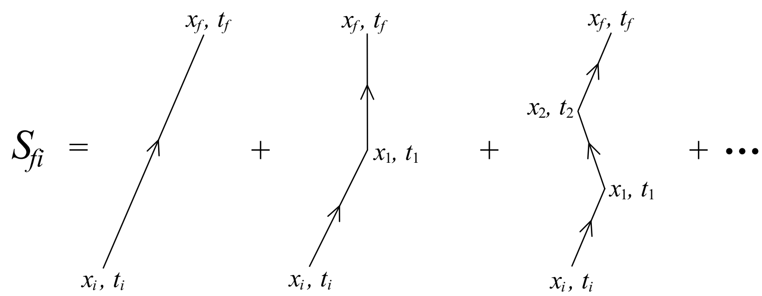

Figure 5: The expansion of S_{fi} in terms of Feynman diagrams

4 Sources, vacuum-to-vacuum transitions and time-ordered products

We now consider a alteration of the system Lagrangian that models the presence of a time-dependent "source". Our discussion follows section 5.5 of Ryder [1] and chapters 1 and 2 of Brown [3]. In this context, we call any external agent that may cause a non-relativistic system to make a transition from one energy eigenstate to another a "source". For example, a time-dependent electric field may induce a charged particle in a one dimensional harmonic oscillator potential to go from one eigenenergy to another. In the context of field theory, a time-dependent source may result in spontaneous particle creation[4]. In either case, the source can be modeled by altering the Lagrangian such that

$$L(q, \dot{q}) \to L(q, \dot{q}) + J(t)q(t). \tag{50}$$

The source $J(t)$ will be assumed to be non-zero in a finite interval $t \in [t_1, t_2]$. We take $T_2 > t_2$ and $T_1 < t_1$. Given that the particle was in it's ground state at $T_1 \to -\infty$, what is the amplitude that the particle will still be in the ground state at time $T_2 \to \infty$?

To answer that question, consider

$$\begin{aligned}
\langle Q_2, T_2 | Q_1, T_1 \rangle_J &= \int dq_1\, dq_2 \langle Q_2, T_2 | q_2, t_2 \rangle \langle q_2, t_2 | q_1, t_1 \rangle_J \langle q_1, t_1 | Q_1, T_1 \rangle \\
&= \int dq_1\, dq_2 \langle Q_2 | e^{-iHT_2} e^{iHt_2} | q_2 \rangle \langle q_2, t_2 | q_1, t_1 \rangle_J \\
&\quad \times \langle q_1 | e^{-iHt_1} e^{iHT_1} | Q_1 \rangle \\
&= \sum_{mn} \int dq_1\, dq_2 \langle Q_2 | e^{-iHT_2} | m \rangle \langle m | e^{iHt_2} | q_2 \rangle \langle q_2, t_2 | q_1, t_1 \rangle_J \\
&\quad \times \langle q_1 | e^{-iHT_1} | n \rangle \langle n | e^{iHt_1} | Q_1 \rangle
\end{aligned}$$

[4] *cf.* PHYS 703 March 14, 2000 lecture

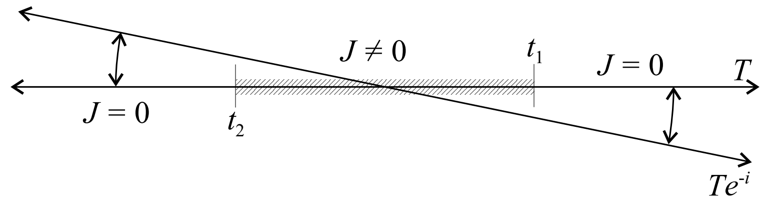

Figure 6: The rotation of the time axis needed to isolate the ground state contribution to the propagator

$$= \sum_{mn} e^{-i(E_n T_2 - E_m T_1)} \phi_m(Q_2)\phi_n^*(Q_1) \times \int dq_1\, dq_2\, \phi_m^*(q_2,t_2)\langle q_2,t_2|q_1,t_1\rangle_J \phi_n(q_1,t_1),$$

where we have introduced a basis of energy eigenstates $H|n\rangle = E_n|n\rangle$ and energy eigenfunctions $\phi_n(q,t) = e^{-iE_n t}\langle q|m\rangle$ with $\phi_n(q) = \langle q|n\rangle$. The J subscripts on the propagators remind us that the source is to be accounted for. It is important to note that $\phi_n(q)$ is only a true eigenfunction for times when the source is not acting; i.e. prior to t_1 and later than t_2. The integral on the last line can be thought of as a wavefunction, $\phi_n(q_1,t_1)$, that is propagated through the time when the source is acting by $\langle q_2,t_2|q_1,t_1\rangle_J$, and is then dotted with a wavefunction $\phi_m^*(q_2,t_2)$. But, $\phi_n(q_1,t_1)$ and $\phi_m^*(q_2,t_2)$ are energy eigenfunctions for times before and after the source, respectively. Hence, the integral is the amplitude that an energy eigenstate $|n\rangle$ will become an energy eigenstate $|m\rangle$ through the action of the source. Now, let's perform a rotation of the time-axis in the complex plane by some small angle $-\delta$ ($\delta > 0$), as shown in figure 6. Under such a transformation

$$T_1 \to T_1 + i|T_1|\delta \tag{51}$$

$$T_2 \to T_2 - i|T_2|\delta, \tag{52}$$

where we have chosen the axis of rotation to lie between T_1 and T_2. We see that the exponential term $e^{-i(E_n T_2 - E_m T_1)}$ will acquire a damping that goes like $e^{-\delta(E_n|T_2|+E_m|T_1|)}$. As we push $T_1 \to -\infty$ and $T_2 \to \infty$, the damping will become infinite for each term in the sum, except for the ground state which we can set to have an energy of $E_0 \geq 0$. Therefore,

$$\lim_{T\to\infty e^{-i\delta}} \langle Q_2,T|Q_1,-T\rangle_J = e^{-iE_0(T_2-T_1)}\phi_0(Q_2)\phi_0^*(Q_1) \times \int dq_1\, dq_2\, \phi_0^*(q_2,t_2)\langle q_2,t_2|q_1,t_1\rangle_J \phi_0(q_1,t_1), \quad \delta > 0, \tag{53}$$

where we have set $T_2 = -T_1 = T$ for convenience. Now, if we take t_2 and t_1 to $\pm\infty$ respectively, the integral reduces to the amplitude that a wavefunction which has the form of $\phi_0(q)$ in the distant past will still have the form of $\phi_0(q)$ in the distant future. In other words, it is the ground-to-ground state transition amplitude, which we denote by

$$\langle 0,\infty|0,-\infty\rangle_J \propto \lim_{T\to\infty\, e^{-i\delta}} \langle Q_2,T|Q_1,-T\rangle_J, \tag{54}$$

where the constant of proportionality depends on Q_1, Q_2 and T. Now, instead of rotating the contour of the the time-integration, we could have added a small term $-i\epsilon q^2/2$ to the Hamiltonian. Using first order perturbation theory, this shifts the energy levels by an amount $\delta E_n = -i\epsilon\langle n|q^2|n\rangle/2$. For most problems (i.e. the harmonic oscillator, hydrogen atom), the expectation value of q^2 increases with increasing energy. Assuming that this is the case for the problem we are doing, we see that the first order shift in the eigenenergy accomplishes the same thing as the rotation of the time axis in (54). But, subtracting $i\epsilon q^2/2$ from H is the same thing as adding $i\epsilon q^2/2$ from L[5]. Therefore,

$$\langle 0,\infty|0,-\infty\rangle_J \propto \int [dQ] \exp\left\{ i\int_{-\infty}^{\infty} dt \left[L(Q,\dot{Q}) + JQ + \frac{1}{2}i\epsilon q^2 \right]\right\}. \tag{55}$$

Finally, want to normalize this result such that if the source is turned off, the amplitude $\langle 0,\infty|0,-\infty\rangle$ is unity. Defining

$$Z[J] = \frac{\int [dQ] \exp\left\{ i\int_{-\infty}^{\infty} dt \left[L(Q,\dot{Q}) + JQ + \frac{1}{2}i\epsilon Q^2 \right]\right\}}{\int [dQ] \exp\left\{ i\int_{-\infty}^{\infty} dt \left[L(Q,\dot{Q}) + \frac{1}{2}i\epsilon Q^2 \right]\right\}}, \tag{56}$$

we have

$$\langle 0,\infty|0,-\infty\rangle_J = Z[J]. \tag{57}$$

Before moving on the the next section, we would like to establish a result that will prove very useful later when we consider field theories. We first define the functional derivative of $Z[J]$ with respect to $J(t')$. Essentially, the functional derivative of a functional $f[y]$, where $y = y(x)$, is the derivative of the discrete expression with respect to the value of y at a given x. For example, the discrete version of $Z[J]$ is

$$Z\left(\tau J(t_0), \tau J(t_1) \ldots \tau J(t_{n-1})\right) \propto \int \exp\left\{ i\tau \sum_{j=0}^{n-1} \left[L(Q_j,\dot{Q}_j) \right.\right. \\ \left.\left. +\ J(t_j)Q_j + \frac{1}{2}i\epsilon Q_j^2 \right]\right\} \prod_{i=1}^{n-1} dQ_i, \tag{58}$$

where we have indicated that the discrete version of $Z[J]$ is an ordinary function of n variables $\tau J(t_j)$ and omitted the normalization factor. We have explicitly included

[5] An alternative procedure for singling out the ground state contribution comes from considering t to be purely imaginary, i.e. consideration of Euclidean space. This is discussed in the next section.

the weighting factor τ with each of the discrete variables to account for the fact that as $n \to \infty$, each $J(t_k)$ covers a smaller and smaller portion of the integration interval. The functional derivative of $Z[J]$ with respect to $J(t_k)$ is then the partial derivative of the discrete expression with respect to $\tau J(t_k)$. Going back to the continuum limit, we write the functional derivative of $Z[J]$ with respect to $Q(t_1)$ as:

$$\frac{\delta Z[J]}{\delta J(t_1)} \propto i \int [dQ] Q(t_1) \exp\left\{ i \int_{-\infty}^{\infty} dt \left[L(Q, \dot{Q}) + JQ + \frac{1}{2} i\epsilon Q^2 \right] \right\}. \tag{59}$$

In a similar fashion, we have

$$\begin{aligned} \frac{\delta^n Z[J]}{\delta J(t_1) \cdots \delta J(t_n)} &\propto i^n \int [dQ] Q(t_1) \cdots Q(t_n) \\ &\quad \times \exp\left\{ i \int_{-\infty}^{\infty} dt \left[L(Q, \dot{Q}) + JQ + \frac{1}{2} i\epsilon Q^2 \right] \right\}. \end{aligned} \tag{60}$$

We notice a similarity between this expression and the expression from statistical mechanics that gives the average value of a microscopic variable in the canonical ensemble. We argue that equation (60) gives the exact same thing: the expectation value of $Q(t_1) \cdots Q(t_n)$. There is one wrinkle, however, which we now proceed to outline. Consider, with $t_k > t_{k'}$,

$$\begin{aligned} \langle q_f, t_f | q(t_k) q(t_{k'}) | q_i, t_i \rangle &= \int dq_1 \cdots dq_{n-1} \langle q_f, t_f | q_{n-1}, t_{n-1} \rangle \cdots \\ &\quad \langle q_k, t_k | q(t_k) | q_{k-1}, t_{k-1} \rangle \cdots \langle q_{k'}, t_{k'} | q(t_{k'}) | q_{k'-1}, t_{k'-1} \rangle \\ &\quad \cdots \langle q_1, t_1 | q_i, t_i \rangle \\ &= \int dq_1 \cdots dq_{n-1} \, q(t_k) q(t_{k'}) \langle q_f, t_f | q_{n-1}, t_{n-1} \rangle \cdots \\ &\quad \langle q_1, t_1 | q_i, t_i \rangle. \end{aligned} \tag{61}$$

By pushing $t_i \to -\infty \, e^{-i\delta}$ and $t_f \to \infty \, e^{-i\delta}$, we can repeat our previous manipulations to show that

$$\begin{aligned} &\int dq_1 \cdots dq_{n-1} \, q(t_k) q(t_{k'}) \langle q_f, \infty \, e^{-i\delta} | q_{n-1}, t_{n-1} \rangle \cdots \langle q_1, t_1 | q_i, -\infty \, e^{-i\delta} \rangle \\ &\qquad = N \int [dq] q(t_k) q(t_{k'}) \exp\left[i \int_{-\infty \, e^{-i\delta}}^{\infty \, e^{-i\delta}} dt \, L(Q, \dot{Q}) \right], \end{aligned} \tag{62}$$

which follows directly from the arguments of section 2, and

$$\langle q_f, \infty \, e^{-i\delta} | q(t_k) q(t_{k'}) | q_i, -\infty \, e^{-i\delta} \rangle \propto \langle 0 | q(t_k) q(t_{k'}) | 0 \rangle, \tag{63}$$

which follows from an argument similar to the one we used to calculate the matrix element $\langle 0, \infty | 0, -\infty \rangle_J$ with $t_1 = t_{k'}$, $t_2 = t_k$ and $J = 0$. Just as before, the rotation of the time axis can be achieved by adding $i\epsilon q^2/2$ to the Lagrangian. This calculation cannot be repeated for the case $t_k > t_{k'}$ because the order of $q(t_k)q(t_{k'})$ in $\langle q_f, t_f|q(t_k)q(t_{k'})|q_i, t_i\rangle$ cannot be switched without introducing terms involving the commutator of H and q. But, in order to perform the decomposition (61), we need the late q operator appearing to the left of the earlier q operator. What this means is that we must be considering the time-ordered product of $q(t_k)$ and $q(t_{k'})$. Putting all of this together along with the expression for the functional derivatives of $Z[J]$, we get

$$\left.\frac{\delta^2 Z[J]}{\delta J(t_k)\delta J(t_{k'})}\right|_{J=0} = i^2\langle 0|T[q(t_k)q(t_{k'})]|0\rangle. \tag{64}$$

This is easily generalized to the time order product of many q operators:

$$\left.\frac{\delta^n Z[J]}{\delta J(t_1)\cdots\delta J(t_n)}\right|_{J=0} = i^n\langle 0|T[q(t_1)\cdots q(t_n)]|0\rangle. \tag{65}$$

We have demanded strict equality in these expression to ensure that the $n = 0$ case returns $\langle 0|0\rangle = 1$. This is a very important formula for what follows.

5 Free scalar fields

We now move on to the quantum field theory of a scalar field $\phi(x)$. In this section we draw on sections 6.1 and 6.3 of Ryder [1], chapter 2 of Popov [4] and section 3.2 of Brown [3]. The classical field ϕ is assumed to satisfy the Klein-Gordon equation

$$(\Box + m^2)\phi = 0. \tag{66}$$

We *define* the vacuum to vacuum transition probability for this theory by

$$\langle 0, \infty|0, -\infty\rangle_J = Z[J], \tag{67}$$

where

$$Z[J] = \frac{1}{Z_0}\int [d\phi]\exp\left\{i\int d^4x\left[\mathcal{L}(\phi) + J(x)\phi + \frac{1}{2}i\epsilon\phi^2\right]\right\}, \tag{68}$$

with

$$Z_0 = \int [d\phi]\exp\left\{i\int d^4x\left[\mathcal{L}(\phi) + \frac{1}{2}i\epsilon\phi^2\right]\right\}. \tag{69}$$

In this expressions, the measure $[d\phi]$ is meant to convey an integration over all field configurations, which can be achieved in practice by dividing spacetime into N^4 points (t_m, x_i, y_j, z_k), with $m, i, j, k = 1\ldots N$. We can schematically merge all of these indices into a single one $n = 1\ldots N^4$. The field is considered to be a collection of N^4 independent variables, and the functional integration $[d\phi]$ becomes $\int\prod_i d\phi_i$. The functional $Z[J]$ represents the *fundamental object in the theory.* Knowledge of the form of $Z[J]$ allows us to derive all the results that we could hope to obtain from experiments involving the field ϕ. Such grandiose statements need to be justified, which we now proceed to do.

We first substitute the Lagrangian appropriate to the Klein-Gordon equation into the expression for $Z[J]$:

$$Z[J] = \frac{1}{Z_0} \int [d\phi] \exp\left(i \int d^4x \left\{ \frac{1}{2} \left[\partial_\alpha \phi \partial^\alpha \phi - (m^2 - i\epsilon)\phi^2 \right] + \phi J \right\} \right). \tag{70}$$

Integrating the $\partial_\alpha \phi \partial^\alpha \phi$ term by parts and using Gauss' theorem to discard the boundary term (assuming $\phi \to 0$ at infinity) gives

$$Z[J] = \frac{1}{Z_0} \int [d\phi] \exp\left(-i \int d^4x \left\{ \frac{1}{2} \phi(\Box + m^2 - i\epsilon)\phi - \phi J \right\} \right). \tag{71}$$

Regarding ϕ as an integration variable, we can change variables according to

$$\phi(x) \to \phi(x) + \phi_0(x). \tag{72}$$

Noting that

$$\int d^4x \, \phi \Box \phi_0 = \int d^4x \, \phi_0 \Box \phi \tag{73}$$

using Gauss' theorem and demanding that $\phi_0 \to 0$ at infinity, we obtain

$$\begin{aligned} Z[J] = \frac{1}{Z_0} \int [d\phi] \exp\Bigg(-i \int d^4x \Bigg\{ & \frac{1}{2} \phi(\Box + m^2 - i\epsilon)\phi \\ & + \phi(\Box + m^2 - i\epsilon)\phi_0 + \frac{1}{2}\phi_0(\Box + m^2 - i\epsilon)\phi_0 - (\phi + \phi_0)J \Bigg\} \Bigg). \end{aligned} \tag{74}$$

Let us demand that

$$(\Box + m^2 - i\epsilon)\phi_0 = J(x). \tag{75}$$

We can solve this equation by introducing the Feynman propagator, which satisfies

$$(\Box + m^2 - i\epsilon)\Delta_F(x) = -\delta^4(x). \tag{76}$$

Hence,

$$\phi_0 = -\int \Delta_F(x - y) J(y) \, d^4y. \tag{77}$$

We then obtain for $Z[J]$:

$$\begin{aligned} Z[J] = & \frac{1}{Z_0} \exp\left[-\frac{i}{2} \int J(x) \Delta_F(x - y) J(y) \, d^4x \, d^4y \right] \\ & \times \int [d\phi] \exp\left[-\frac{i}{2} \int \phi(\Box + m^2 - i\epsilon)\phi \, d^4x \right]. \end{aligned} \tag{78}$$

But we know $Z[0] = 1$, so we must have

$$Z_0 = \int [d\phi] \exp\left[-\frac{i}{2}\int \phi(\Box + m^2 - i\epsilon)\phi\, d^4x\right]. \tag{79}$$

This leads to our final expression:

$$Z[J] = \exp\left[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\, d^4x\, d^4y\right]. \tag{80}$$

Before we move forward, we would like to make a comment on the inclusion of the $i\epsilon\phi^2/2$ term in our expression for $Z[J]$. The reader will recall that the reason that we added this term was to simulate the effects of rotating the time axis by a small angle $-\delta$ in the complex plane. Instead of adding the ϵ-term, we can instead rotate the t axis by $-\pi/2$ so that $t = -i\tau$. The metric becomes $\eta^{\alpha\beta} = \mathrm{diag}(-1,-1,-1,-1)$, which means that we are considering a Euclidean, not Lorentzian, manifold. In that case, the vacuum-to-vacuum transition amplitude is

$$Z[J] = \frac{1}{Z_0}\int [d\phi] \exp\left(-\int d\tau d\mathbf{x}\left\{\frac{1}{2}\left[(\partial_\tau\phi)^2 + \nabla\phi^2 + m^2\phi^2\right] - \phi J\right\}\right), \tag{81}$$

where ∇ is the del-operator of 3D vector calculus. For $J = 0$, the argument of the exponential is negative definite and the integral converges absolutely. So, in some sense, the rotation of the time axis ensures that the integrals in $Z[J]$ are well behaved. Another aspect the Euclidean manifold comes from the Euclidean version of Feynman propagator, which satisfies:

$$(\partial_\tau^2 + \nabla^2 - m^2)\Delta_F(x) = \delta^4(x). \tag{82}$$

Solving this in Euclidean Fourier space, we get

$$\Delta_F(x) = -\frac{i}{(2\pi)^4}\int d^4\kappa \frac{e^{-i\kappa\cdot x}}{\kappa^2 + m^2}, \tag{83}$$

where $\kappa \cdot x = \kappa^0\tau + \mathbf{k}\cdot\mathbf{x}$ and $\kappa^2 = (\kappa^0)^2 + \mathbf{k}^2$ (the i is there for convenience). If we change variables according to $\kappa_0 = -ik_0$ and $\tau = it$, we get

$$\Delta_F(x) = -\frac{1}{(2\pi)^4}\int_{C'} d^4k \frac{e^{-ik\cdot x}}{k^2 - m^2}, \tag{84}$$

where the k_0 integration is to be performed along the contour C' shown in figure 7. But, we can rotate the contour by 90^o clockwise to C, which is the standard contour used to calculate the Lorentzian Feynman propagator, defined by (76). To some extent, this explains why the Feynman propagator has the form that it does: it is a direct consequence of the need to rotate the time axis to isolate the vacuum contributions to transition amplitudes and make path integrals converge.

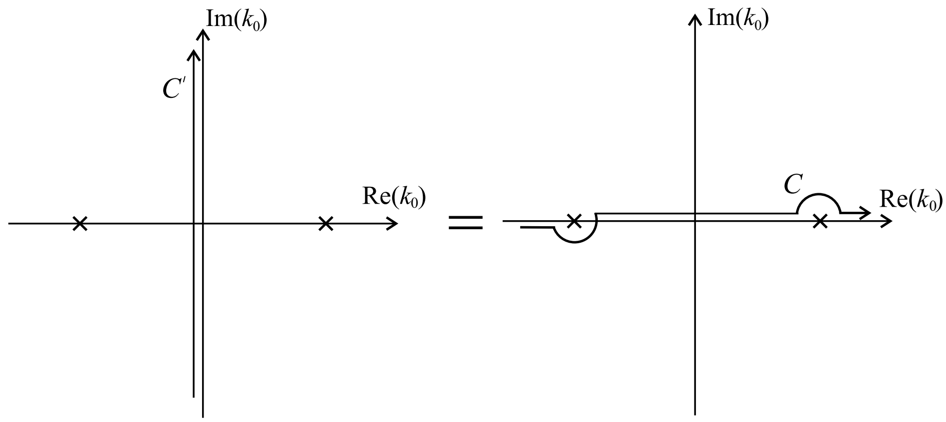

Figure 7: Equivalent contours of integration for the calculation of Feynman propagator in the complex k_0 plane. Note the poles at $\pm\omega = \pm\sqrt{|\mathbf{k}|^2 + m^2}$.

We want to calculate the time-ordered products given by

$$\frac{1}{i^n} \frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)}\bigg|_{J=0} = \langle 0|T[\phi(x_1) \cdots \phi(x_n)]|0\rangle. \tag{85}$$

We recall from the canonical formulation of field theory that $\langle 0|T[\phi(x)\phi(y)]|0\rangle$ is the amplitude for the creation of a particle at y and its later destruction at x (or vice versa, depending on the times associated with x and y). Using (80) and our previously mentioned notions of functional differentiation, we have

$$\frac{1}{i} \frac{\delta}{\delta J(x_1)} Z[J] = -Z[J] \int dy\, \Delta_F(x_1 - y) J(y). \tag{86}$$

The second order derivative is

$$\begin{aligned} \frac{1}{i} \frac{\delta}{\delta J(x_2)} \frac{1}{i} \frac{\delta}{\delta J(x_1)} Z[J] &= Z[J] \int dy_1\, \Delta_F(x_1 - y_1) J(y_1) \int dy_2\, \Delta_F(x_2 - y_2) J(y_2) \\ &+ i\Delta_F(x_1 - x_2) Z[J]. \end{aligned}$$

Continuing,

$$\begin{aligned} &\frac{1}{i} \frac{\delta}{\delta J(x_3)} \frac{1}{i} \frac{\delta}{\delta J(x_2)} \frac{1}{i} \frac{\delta}{\delta J(x_1)} Z[J] = \\ &- i\Delta_F(x_1 - x_2) Z[J] \int dy_3\, \Delta_F(x_3 - y_3) J(y_3) \\ &- i\Delta_F(x_1 - x_3) Z[J] \int dy_2\, \Delta_F(x_2 - y_2) J(y_2) \\ &- i\Delta_F(x_2 - x_3) Z[J] \int dy_1\, \Delta_F(x_1 - y_1) J(y_1) \end{aligned}$$

$$
\begin{aligned}
&- Z[J] \int dy_1\, dy_2\, dy_3\, \Delta_F(x_1 - y_1) J(y_1) \\
&\times \Delta_F(x_2 - y_2) J(y_2) \Delta_F(x_3 - y_3) J(y_3)
\end{aligned}
$$

Finally, we write down the fourth order derivative:

$$
\begin{aligned}
\frac{1}{i}\frac{\delta}{\delta J(x_4)}\frac{1}{i}\frac{\delta}{\delta J(x_3)}\frac{1}{i}\frac{\delta}{\delta J(x_2)}\frac{1}{i}\frac{\delta}{\delta J(x_1)} Z[J] &= i\Delta_F(x_1 - x_2) i\Delta_F(x_3 - x_4) Z[J] \\
&+ i\Delta_F(x_1 - x_3) i\Delta_F(x_2 - x_4) Z[J] \\
&+ i\Delta_F(x_2 - x_3) i\Delta_F(x_1 - x_4) Z[J] \\
&+ \text{ other terms that vanish when } J = 0.
\end{aligned}
$$

When $J = 0$, these expression give us the following time-ordered products:

$$
\begin{aligned}
\langle 0|T[\phi(x_1)]|0\rangle &= 0 && (87) \\
\langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle &= i\Delta_F(x_1 - x_2) && (88) \\
\langle 0|T[\phi(x_1)\phi(x_2)\phi(x_3)]|0\rangle &= 0 && (89) \\
\langle 0|T[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)]|0\rangle &= i\Delta_F(x_1 - x_2) i\Delta_F(x_3 - x_4) \\
&+ i\Delta_F(x_1 - x_3) i\Delta_F(x_2 - x_4) \\
&+ i\Delta_F(x_1 - x_4) i\Delta_F(x_2 - x_3) && (90)
\end{aligned}
$$

We call $\langle 0|T[\phi(x_1)\cdots\phi(x_n)]|0\rangle$ an n-point correlation function. Generalizing the pattern above to T products of more field operators, we find that if n is odd, the n-point function vanishes. On the other hand, if n is even, the correlation function reduces to the sum all possible permutations of products of 2-point functions $i\Delta_F(x - y)$ with $x, y \in \{x_1, \ldots, x_n\}$, $x \neq y$. When this result is derived from canonical methods, it is know as *Wick's Theorem.*

It is interesting to interpret these results in terms of the Taylor series expansion of $Z[J]$ about $J = 0$. To make sense of this object, recall that the Taylor series expansion of a function of a finite number of variables is

$$
F(y_1, \ldots, y_k) = \sum_{n=0}^{\infty} \sum_{i_1=1}^{k} \cdots \sum_{i_n=1}^{k} \frac{1}{n!} y_{i_1} \ldots y_{i_k} \left. \frac{\partial^n F}{\partial y_{i_1} \ldots \partial y_{i_n}} \right|_{y_i=0}. \tag{91}
$$

This is expansion is taken about the zero of all of the independent variables. Assuming the variables are weighted appropriately, when we go to the continuum limit $k \to \infty$ we obtain that

$$
F[y] = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \cdots dx_n\, y(x_1) \cdots y(x_n) \left. \frac{\delta^n F}{\delta y(x_1) \ldots \delta y(x_n)} \right|_{y=0}. \tag{92}
$$

Using this expansion, we obtain

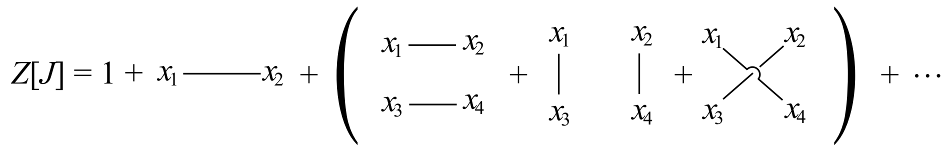

Figure 8: Pictorial representation of the free field generating functional $Z[J]$

$$
\begin{aligned}
Z[J] &= 1 + \frac{i^2}{2!} \int dx_1\, dx_2\, J(x_1)J(x_2) i\Delta_F(x_1 - x_2) \\
&+ \frac{i^4}{4!} \int dx_1\, dx_2\, dx_3\, dx_4 J(x_1)J(x_2)J(x_3)J(x_4) \\
&\times \Big[i\Delta_F(x_1 - x_2) i\Delta_F(x_3 - x_4) + i\Delta_F(x_1 - x_3) i\Delta_F(x_2 - x_4) \\
&+ i\Delta_F(x_1 - x_4) i\Delta_F(x_2 - x_3) \Big] + \cdots
\end{aligned}
\tag{93}
$$

This result is depicted diagrammatically in figure 8. In this figure, we use the following Feynman rules:

1. Each line is connected to a sink and a source.
2. Each sink or a source is labeled with a spacetime point.
3. A line running between x and y comes with a propagator $i\Delta_F(x - y)$.
4. Each source or sink attached to x comes with a factor $iJ(x)$.
5. The collection of all graphs with n sources and sinks is multiplied by $1/n!$.
6. All spacetimes coordinates are integrated over.

This first rule ensures that we only include diagrams with an even number of vertices and that we only consider disconnected graphs. The Feynman diagrams illustrate the correspondence of $Z[J]$ to the vacuum-to-vacuum transition probability beautifully. Each term in the graph involves the creation of a particle at some point and its destruction at a later point. The three terms in the 4-vertex graph account for all possible permutations of particle being created/destroyed at a pair of x_1, x_2, x_3 or x_4. What we would like to do now is move on to the more interesting case of interacting fields.

6 The generating functional for self-interacting fields

While free field theory has a certain amount of elegance to it, it is not terribly interesting. In this section, we consider a self-interacting field whose Lagrangian is given by

$$\mathcal{L}(\phi) = \frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi - \frac{1}{2}m^2\phi^2 + \mathcal{L}_{\rm int}(\phi) = \mathcal{L}_0(\phi) + \mathcal{L}_{\rm int}(\phi). \tag{94}$$

The discussion follows section 6.4 of Ryder [1]. Here, $\mathcal{L}_{\rm int}(\phi)$ is the Lagrangian describing the self interaction. The generating functional is

$$Z[J] = \frac{\int [d\phi] \exp\left(iS + i\int d^4x\, J\phi\right)}{\int [d\phi] e^{iS}}, \tag{95}$$

where S is the classical action

$$S = \int d^4x\, [\mathcal{L}_0(\phi) + \mathcal{L}_{\rm int}(\phi)] = S_0 + S_{\rm int}. \tag{96}$$

We have dropped the $i\epsilon\phi^2/2$ term used to single out the ground state, which we can rationalize by the rotation of the time axis. In this section, we will write the free field propagator as

$$Z_0[J] = \frac{\int [d\phi] \exp\left(iS_0 + i\int d^4x\, J\phi\right)}{\int [d\phi] e^{iS_0}}. \tag{97}$$

We would like to write $Z[J]$ in a form particularly useful for calculations. We can't really reproduce the manipulations of the last section because the $\mathcal{L}_{\rm int}$ in the action S introduces difficulties when the shift $\phi \to \phi + \phi_0$ is performed. We will instead derive a differential equation satisfied by $Z[J]$ and then solve it in terms of $J(x)$ and the Feynman propagator. The result will be something that we might have guessed intuitively. What is the equation satisfied by the free field propagator? Now, we know that

$$\frac{1}{i}\frac{\delta}{\delta J(x)} Z_0[J] = -Z_0[J] \int dy\, \Delta_F(x-y) J(y). \tag{98}$$

We operate on both sides with $\Box_x + m^2$ and use the defining relation for the Feynman propagator (76) to get:

$$(\Box_x + m^2)\frac{1}{i}\frac{\delta}{\delta J(x)} Z_0[J] = J(x) Z_0[J]. \tag{99}$$

The is the differential equation satisfied by $Z_0[J]$. In order to find the differential equation satisfied by $Z[J]$, let us define the functional

$$\hat{Z}[\phi] = \frac{e^{iS}}{\int [d\phi]\, e^{iS}}. \tag{100}$$

Then

$$Z[J] = \int [d\phi]\, \hat{Z}[\phi] \exp\left(i \int d^4x\, J\phi\right), \tag{101}$$

which is the functional equivalent of the Fourier transform. We can functionally differentiate $\hat{Z}[\phi]$ with respect to ϕ using

$$
\begin{aligned}
S &= \int d^4x \left[\frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi - \frac{1}{2}m^2\phi^2 + \mathcal{L}_{\rm int}(\phi)\right] \\
&= -\int d^4x \left[\frac{1}{2}\phi(\Box + m^2)\phi - \mathcal{L}_{\rm int}\right],
\end{aligned} \tag{102}
$$

where Gauss' theorem has been used. The superiority of the latter form is that we can integrate twice by parts to change $\phi\overrightarrow{\Box}\phi$ into $\phi\overleftarrow{\Box}\phi$. So, when we functionally differentiate, we can make sure that the ϕ being acted on by the $\Box$ operator is different from the ϕ being acted on by the $\delta/\delta\phi$. We get

$$
i\frac{\delta\hat{Z}[\phi]}{\delta\phi(x)} = (\Box + m^2)\phi(x)\hat{Z}[\phi] - \mathcal{L}'_{\rm int}(\phi)\hat{Z}[\phi], \tag{103}
$$

where

$$
\mathcal{L}'_{\rm int}(\phi) = \frac{\partial\mathcal{L}_{\rm int}}{\partial\phi} \tag{104}
$$

and $\Box = \Box_x$. We multiply both sides of (103) by $\exp[i\int J(y)\phi(y)\,d^4y]$ and integrate over ϕ, i.e. we take the Fourier transform. The RHS of (103) becomes

$$
\begin{aligned}
\text{RHS} = {} & \frac{1}{Z_0}(\Box + m^2)\int[d\phi]\phi(x)\exp\left[iS + i\int J(y)\phi(y)\,d^4y\right] \\
& - \frac{1}{Z_0}\int[d\phi]\mathcal{L}'_{\rm int}(\phi)\exp\left[iS + i\int J(y)\phi(y)\,d^4y\right],
\end{aligned} \tag{105}
$$

where we have written

$$
Z_0 = \int[d\phi]e^{iS}. \tag{106}
$$

Now, it's easy to see

$$
\frac{1}{i}\frac{\delta Z[J]}{\delta J(x)} = \frac{1}{Z_0}\int[d\phi]\,\phi(x)\exp\left[iS + i\int J(y)\phi(y)\,d^4y\right], \tag{107}
$$

which leads to

$$
\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^n Z[J] = \frac{1}{Z_0}\int[d\phi]\,\phi^n(x)\exp\left[iS + i\int J(y)\phi(y)\,d^4y\right]. \tag{108}
$$

Now, we assume that $\mathcal{L}'_{\rm int}(\phi)$ possesses a Taylor series expansion in ϕ. We can then reproduce the second term in (105) by adding together a series of contributions of the form of (108). Hence,

$$
\text{RHS} = (\Box + m^2)\frac{1}{i}\frac{\delta Z[J]}{\delta J(x)} - \mathcal{L}'_{\rm int}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)Z[J]. \tag{109}
$$

The Fourier transform of the LHS of (103) becomes

$$\begin{aligned}
\text{LHS} &= i\int [d\phi] \frac{\delta \hat{Z}(\phi)}{\delta\phi(x)} \exp\left[i\int J(y)\phi(y)\, d^4y\right] \\
&= i\hat{Z}(\phi)\exp\left[i\int J(y)\phi(y)\, d^4y\right]\Big|_{\phi\to\infty} \\
&\quad -i\int [d\phi]\hat{Z}(\phi)\frac{\delta}{\delta\phi(x)}\exp\left[i\int J(y)\phi(y)\, d^4y\right] \\
&= \int [d\phi] J(x)\hat{Z}(\phi)\exp\left[i\int J(y)\phi(y)\, d^4y\right] \\
&= J(x)Z[J]. \qquad (110)
\end{aligned}$$

In the second line we have performed a functional integration by parts, which follows from the fact that the functional derivative satisfies the product rule and functional integral obeys the fundamental theorem of calculus. The boundary term must vanish as $\phi \to \infty$ because if it didn't, the integral for $Z[J]$ would diverge. Putting together our formulae for the LHS and RHS of the Fourier transform of (103):

$$(\Box + m^2)\frac{1}{i}\frac{\delta Z[J]}{\delta J(x)} - \mathcal{L}'_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right) Z[J] = J(x)Z[J]. \qquad (111)$$

This is the differential equation satisfied by $Z[J]$. We see that if the interacting Lagrangian is set to zero, the result reduces to (99).

We will assume a solution of the differential equation of the form

$$Z[J] = N\exp\left[i\int d^4x\, \mathcal{L}_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J}\right)\right] Z_0[J] \qquad (112)$$

partly because this is what we might expect from $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$ if we replace $-i\delta/\delta J$ with ϕ, as we usually do, partly because we know it's the right answer. As usual, N is a normalizing factor. Let's first establish and identity:

$$\begin{aligned}
\exp\left[-i\int d^4y\, \mathcal{L}_{\text{int}}\left(\tfrac{1}{i}\tfrac{\delta}{\delta J(y)}\right)\right] J(x) \exp\left[i\int d^4y\, \mathcal{L}_{\text{int}}\left(\tfrac{1}{i}\tfrac{\delta}{\delta J(y)}\right)\right] \\
= J(x) - \mathcal{L}'_{\text{int}}\left(\tfrac{1}{i}\tfrac{\delta}{\delta J(x)}\right). \qquad (113)
\end{aligned}$$

The first step is to notice that the commutator of J_i and $\partial/\partial J_k$ when acting on a function of $(J_1, \ldots, J_n)$ is given by the well known relation

$$\left[J_i, \frac{1}{i}\frac{\partial}{\partial J_k}\right] = i\delta_{ik}. \qquad (114)$$

The continuum limit ($n \to \infty$) of this is

$$\left[J(x), \frac{1}{i}\frac{\delta}{\delta J(y)}\right] = i\delta(x-y). \qquad (115)$$

Now, it's not hard to see that if A and B are operators whose commutator is a number a, then

$$\begin{aligned} [A, B] &= a, \\ [A, B^2] &= 2aB, \\ [A, B^3] &= 3aB^2, \\ &\vdots \\ [A, B^n] &= naB^{n-1}. \end{aligned}$$

Hence,

$$\left[J(x), \left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^n\right] = i\delta(x-y)n\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^{n-1}. \tag{116}$$

Assume that $\mathcal{L}_{\text{int}}(\phi)$ possesses a Taylor series expansion

$$\mathcal{L}_{\text{int}}(\phi) = c_0 + c_1\phi + \frac{1}{2!}c_2\phi^2 + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}c_n\phi^n. \tag{117}$$

Since $[A, (B+C)] = [A, B] + [A, C]$, we can write

$$\left[J(x), \sum_{n=0}^{\infty}\frac{1}{n!}c_n\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^n\right] = i\delta(x-y)\sum_{n=1}^{\infty}\frac{1}{(n-1)!}c_n\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)^{n-1}. \tag{118}$$

But

$$\mathcal{L}'_{\text{int}}(\phi) = \sum_{n=0}^{\infty}\frac{n}{n!}c_n\phi^{n-1} = \sum_{n=1}^{\infty}\frac{1}{(n-1)!}c_n\phi^{n-1}. \tag{119}$$

Using this an integrating (118) with respect to y gives

$$\left[J(x), i\int d^4y\,\mathcal{L}_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)\right] = -\mathcal{L}'_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right). \tag{120}$$

where we have scaled $\mathcal{L}_{\text{int}}$ by i. Finally, note that the Hausdorff formula gives

$$e^{-B}Ae^{B} = A + [A, B] \tag{121}$$

when $[A, B]$ is a number. Putting (120) into (121) with $A = J(x)$ and $B = i\int d^4y\,\mathcal{L}_{\text{int}}\,(-i\delta/\delta J(y))$ yields (113).

Using our assumed form for $Z[J]$ (112) and the identity (113) we get

$$\begin{aligned} J(x)Z[J] &= N J(x)\exp\left[i\int d^4y\,\mathcal{L}_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)\right]Z_0[J] \\ &= N\exp\left[i\int d^4y\,\mathcal{L}_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)\right]\left[J(x) - \mathcal{L}'_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right]Z_0[J] \end{aligned}$$

$$= N \exp\left[i \int d^4y\, \mathcal{L}_{\rm int}\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)\right] (\Box_x + m^2) \frac{1}{i}\frac{\delta}{\delta J(x)} Z_0[J]$$
$$-\mathcal{L}'_{\rm int}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right) N \exp\left[i \int d^4y\, \mathcal{L}_{\rm int}\left(\frac{1}{i}\frac{\delta}{\delta J(y)}\right)\right] Z_0[J]$$
$$= (\Box_x + m^2) \frac{1}{i}\frac{\delta Z[J]}{\delta J(x)} - \mathcal{L}'_{\rm int}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right) Z[J].$$

In going from the second to the third line, we have used the differential equation satisfied by $Z_0[J]$ (99). This result is just the differential equation (111), which confirms that we can write

$$Z[J] = \frac{\exp\left[i \int d^4x\, \mathcal{L}_{\rm int}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right] Z_0[J]}{\exp\left[i \int d^4x\, \mathcal{L}_{\rm int}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\right] Z_0[J]\Big|_{J=0}} \tag{122}$$

where we have

$$Z_0[J] = \exp\left[-\frac{i}{2}\int J(x)\Delta_F(x-y)J(y)\, d^4x\, d^4y\right]. \tag{123}$$

With these two equations, we have succeeding in writing down the generating functional entirely in terms of the source J and the Feynman propagator Δ_F.

7 ϕ^4 and ϕ^3 theory

We would like to demonstrate the calculation of the generating functional $Z[J]$ and some n-point functions in the case of self-interacting ϕ^4 and ϕ^3 theory. We follow section 6.5 of Ryder [1] and chapter 2 of Popov [4]. We first consider ϕ^4 theory, which has the interacting Lagrangian:

$$\mathcal{L}_{\rm int}(\phi) = -\frac{\lambda}{4!}\phi^4. \tag{124}$$

It goes without saying that λ is small. The generating functional $Z_4[J]$ for ϕ^4 is

$$Z_4[J] = \frac{\exp\left[-\frac{i\lambda}{4!}\int d^4z\, \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4\right] Z_0[J]}{\exp\left[-\frac{i\lambda}{4!}\int d^4z\, \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^4\right] Z_0[J]\Big|_{J=0}} \tag{125}$$

The numerator of $Z_4[J]$ is, to first order in λ, is:

$$\text{num}\, Z_4[J] = \left[1 - \frac{i\lambda}{4!}\int d^4z\, \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^4 + \cdots\right] Z_0[J]. \tag{126}$$

Performing the functional differentiation, we obtain

$$\text{num}\, Z_4[J] = \left\{1 - \frac{i\lambda}{4!}\int d^4z \left[-3\Delta_F^2(0) + 6i\Delta_F(0)\left(\int d^4x\, \Delta_F(z-x)J(x)\right)^2 \right.\right.$$
$$\left.\left. + \left(\int d^4x\, \Delta_F(z-x)J(x)\right)^4\right] + \cdots\right\} Z_0[J], \tag{127}$$

where we have written $\Delta_F(z-z) = \Delta_F(0)$[6]. We also get

$$\text{denom}\, Z_4[J] = \text{num}\, Z_4[J]\Big|_{J=0} = \left\{1 - \frac{i\lambda}{4!}\int d^4z\, \left[-3\Delta_F^2(0)\right] + \cdots\right\} Z_0[J]. \tag{128}$$

Putting the two results together yields, to order λ, we have:

$$Z_4[J] = \left\{1 - \frac{i\lambda}{4!}\int d^4z \left[6i\Delta_F(0)\left(\int d^4x\, \Delta_F(z-x)J(x)\right)^2 \right.\right.$$
$$\left.\left. + \left(\int d^4x\, \Delta_F(z-x)J(x)\right)^4\right] + \cdots\right\} Z_0[J], \tag{129}$$

Now, let's do the same thing for ϕ^3 theory, where the interacting Lagrangian is

$$\mathcal{L}_{\text{int}}(\phi) = -\frac{\lambda}{3!}\phi^3. \tag{130}$$

The numerator of the generating functional is

$$\text{num}\, Z_3[J] = \exp\left[-\frac{i\lambda}{3!}\int d^4z \left(\frac{1}{i}\frac{\delta}{\delta J(z)}\right)^3\right] Z_0[J]. \tag{131}$$

Expanding to order λ, we get

$$\text{num}\, Z_3[J] = \left\{1 - \frac{i\lambda}{3!}\int d^4z \left[-3i\Delta_F(0)\int d^4x\, \Delta_F(z-x)J(x) \right.\right.$$
$$\left.\left. - \left(\int d^4x\, \Delta_F(z-x)J(x)\right)^3\right] + \cdots\right\} Z_0[J]. \tag{132}$$

If we set $J = 0$, the above reduces to unity. So, the denominator of $Z_3[J]$ is 1, to first order in λ, which yields:

$$Z_3[J] = \left\{1 - \frac{i\lambda}{3!}\int d^4z \left[-3i\Delta_F(0)\int d^4x\, \Delta_F(z-x)J(x) \right.\right.$$
$$\left.\left. - \left(\int d^4x\, \Delta_F(z-x)J(x)\right)^3\right] + \cdots\right\} Z_0[J]. \tag{133}$$

[6]The Feynman propagator evaluated at zero is, of course, infinite. It's inclusion in the generating functional represents the infinite self-energy of particles in the theory and must be regulated by renormalization, which we will not consider here.

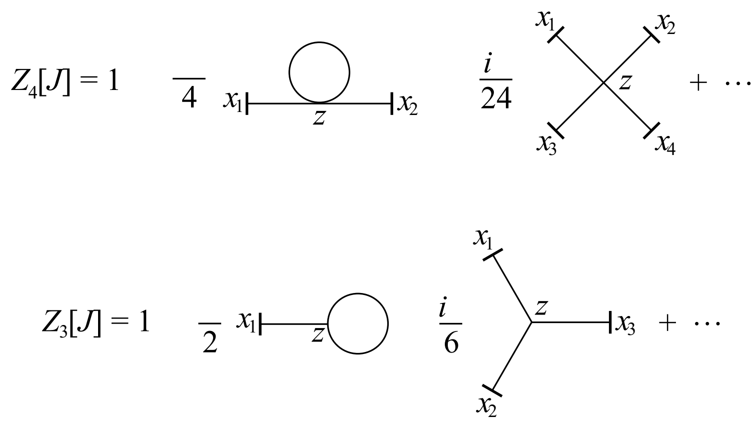

Figure 9: Pictorial representation of the generating functional for ϕ^4 and ϕ^3 theory respectively

We can represent the two generating functionals $Z_4[J]$ and $Z_3[J]$ diagrammatically with the following Feynman rules:

1. The spacetime point z_i is associated with internal points, all other variables go with external points (recall that z_i is the coordinate that occurred in the interacting Lagrangian).

2. A line between x and z comes with a propagator $\Delta_F(x-z)$.

3. Each internal point comes with a factor of $i\lambda/4!$ for ϕ^4 theory, $i\lambda/3!$ for ϕ^3 theory.

4. External points x come with a factor $J(x)$.

5. Terms of the form $\Delta_F(0)$ represent closed loops, or propagators who begin and end at the same point, joined to internal points.

6. All spacetime points are integrated over.

7. Each term in the series is multiplied by the free particle propagator $Z_0[J]$.

These rules give the pictures in figure 9 for $Z_4[J]$ and $Z_3[J]$. These diagrams stress the general structure of the two theories. Because of the power of 4 in the interacting Lagrangian in ϕ^4 theory, the vertices in the associated Feynman diagrams are attached to 4 legs. Similarly, the vertices in ϕ^3 theory are attached to 3 legs. A peculiarity of ϕ^3 theory is the diagram with only one external point. As we will see below, this gives rise to a one point correlation function which means that particles in ϕ^3 can spontaneously self-destruct.

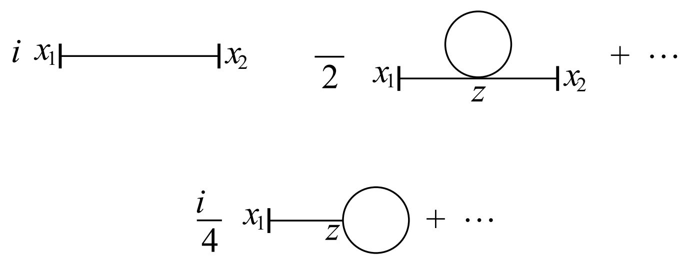

Figure 10: Pictorial representation of the two-point function $\langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle_4$ for ϕ^4 theory (*top*) and of the one-point function $\langle 0|T[\phi(x_1)]|0\rangle_3$ for ϕ^3 theory (*bottom*)

Let's calculate the 2-point function for ϕ^4 theory:

$$\langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle_4 = -\left.\frac{\delta^2 Z_4[J]}{\delta J(x_2)\delta J(x_1)}\right|_{J=0}. \tag{134}$$

The functional differentiation is easy to do from (129) using the our previously derived formulae for derivatives of the free field propagator (87) – (90). The result is

$$\begin{aligned}\langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle_4 &= i\Delta_F(x_1 - x_2) - \frac{\lambda}{2}\Delta_F(0)\int d^4z\, \Delta_F(x_1 - z)\Delta_F(z - x_2) \\ &\quad + O(\lambda^2).\end{aligned} \tag{135}$$

This can be turned into a Feynman diagram, as in figure 10, using rules similar to the ones for $Z[J]$, except now the only the internal points are integrated over. Figure 10 shows how the amplitude for the creation of a particle at point x_1 and its later destruction at x_2 is modified by an (infinite) loop diagram to first-order in λ in ϕ^4 theory.

Now, let's get the 2-point function in ϕ^3 theory. It is simply calculated

$$\langle 0|T[\phi(x_1)\phi(x_2)]|0\rangle_3 = -\left.\frac{\delta^2 Z_3[J]}{\delta J(x_2)\delta J(x_1)}\right|_{J=0} = i\Delta_F(x_1 - x_2) + O(\lambda^2). \tag{136}$$

This isn't terribly interesting, we see that there are no corrections to the free field result to this level in perturbation theory. We shouldn't be surprised, there is no way to connect to external lines to a single vertex if the vertex must be attached to three lines. However, there is a way to connect a vertex to one external line and one internal line, as shown in figure 10. This is borne out by the calculation of the one-point function:

$$\begin{aligned}\langle 0|T[\phi(x_1)]|0\rangle_3 &= -i\left.\frac{\delta^2 Z_3[J]}{\delta J(x_1)}\right|_{J=0} \\ &= \frac{i\lambda}{4}\Delta_F(0)\int d^4z\, \Delta_F(z - x_1) + O(\lambda^2).\end{aligned} \tag{137}$$

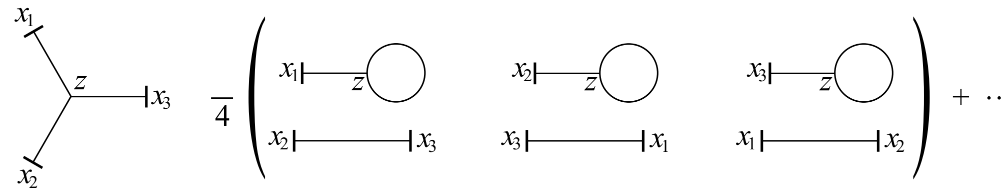

Figure 11: Pictorial representation of $\langle 0|T[\phi(x_1)\phi(x_2)\phi(x_3)]|0\rangle_3$, the three point function, for ϕ^3 theory

Certainly a strange beast, this represents the amplitude of a particle being spontaneously created out of the vacuum, all by itself[7]. We can also calculate the three point function

$$\begin{aligned}\langle 0|T[\phi(x_1)\phi(x_2)\phi(x_3)]|0\rangle_3 &= i\frac{\delta^2 Z_3[J]}{\delta J(x_3)\delta J(x_2)\delta J(x_1)}\Big|_{J=0} \\ &= -\lambda\int d^4z\,\Delta_F(z-x_1)\Delta_F(z-x_2)\Delta_F(z-x_3) \\ &+ \frac{\lambda}{4}\Delta_F(0)\int d^4z\,[\Delta_F(z-x_1)\Delta_F(x_2-x_3) \\ &+ \Delta_F(z-x_2)\Delta_F(x_3-x_1) \\ &+ \Delta_F(z-x_3)\Delta_F(x_1-x_2)]. \end{aligned} \tag{138}$$

The first integral involves all the external legs attached to the same point, i.e. two particles merging into one or one particle splitting in two. The second integral is the product of free 2-point functions and the 1-point function we have already calculated. The diagram is in figure 11.

We have hence shown how the generating functional and n-point functions can be found from simple functional differentiation of equations (122) and (123) and expressed in terms of Feynman diagrams. These diagrams can be converted into scattering amplitudes via the LSZ reduction formula[8]:

$$\begin{aligned}\langle \mathbf{p}_1,\ldots,\mathbf{p}_n,+\infty|\mathbf{q}_1,\ldots,\mathbf{q}_m,-\infty\rangle &= \text{disconnected terms} \\ &+ (i)^{n+m}\int dy_1\cdots dy_n\,dx_1\cdots dx_m \\ &\times e^{i(p_1\cdot y_1+\cdots+p_n\cdot y_n-q_1\cdot x_1-\cdots-q_m\cdot x_m)} \\ &\times (\Box_{y_1}+m^2)\cdots(\Box_{y_n}+m^2) \\ &\times (\Box_{x_1}+m^2)\cdots(\Box_{x_m}+m^2) \\ &\times \langle 0|T[\phi(y_1)\cdots\phi(y_n)\phi(x_1)\ldots\phi(x_m)]|0\rangle.\end{aligned}$$

[7]However, such a process would be forbidden on energy grounds

[8]*cf.* PHYS 703 March 16, 2000. Brown [3] claims that this formula can be obtained via path integral methods, by he presents a proof using the canonical formalism.

In practice, these scattering amplitudes are the only meaningful quantities in quantum field theory since they are the only things that can be directly measured. So, having arrived at a point where we can calculate $\langle \mathbf{p}_1, \ldots, \mathbf{p}_n, +\infty | \mathbf{q}_1, \ldots, \mathbf{q}_m, -\infty \rangle$ using the generating functional, we have completed our formulation of self-interacting field theories in terms of path integrals.

More complicated theories, such as QED, can be quantized in terms of path-integrals, but there are several issues that need to be addressed when writing down $Z[J]$ for gauge fields. One finds that $Z[J]$ is infinite for gauge fields A^α, because the $\int [dA^\alpha]$ integration includes an infinite number of contributions from fields related by a simple gauge transformation. The resolution is the addition of gauge fixing terms to the Lagrangian and the appearance of non-physical "ghost fields". Such things are beyond the scope of this paper.

8 Conclusions

We have demonstrated how quantum mechanics can be formulated in terms of the propagator that represents the amplitude of a particle travels from some initial position to some final position. We have determined the form of the propagator to be a sum over paths of the amplitudes associated with the particle traveling along a given trajectory. The amplitude associated with the path $q(t)$ was shown to be $e^{iS[q]}$, where $S[q]$ is the classical Lagrangian. We used this form of the propagator to derive Feynman rules for the S_{fi} matrix in non-relativistic scattering problems. Then, we showed that by adding a source $J(t)$ to the Lagrangian and rotating the time axis in the complex plane, the propagator $Z[J]$ reduces to the ground state-to-ground state transition amplitude. Functional derivatives of $Z[J]$ with respect to the source $J(t)$ were shown to give the time-ordered product of configuration operators $q(t)$.

We generalized the propagator to free scalar fields and expanded $Z[J]$ in terms of Feynman diagrams by expressing it in terms of the Feynman propagator and the source $J(x)$. In the process, we derived some n-point correlation functions by differentiating $Z[J]$. We discussed how the rotation of the time axis needed to isolate the vacuum-to-vacuum transition probability is directly responsible for the prominent role that the Feynman propagator plays in field theory. We developed a formalism to deal with self-interacting fields and expressed $Z[J]$ entirely in terms of $\Delta_F(x)$, $J(x)$, and the series expansion of the interacting Lagrangian. For both ϕ^4 and ϕ^3 theory, we expressed $Z[J]$ and some n-point functions in terms of Feynman diagrams. Since the scattering problem is essentially solved once the procedure for calculating n-point functions is specified (ignoring issues of renormalization), we have hence shown how all of the theory of self-interacting scalar fields can be derived from path integrals.

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