Gravitational waves from braneworld black holes: the black string braneworld

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Abstract In these lecture notes, we present the black string model of a braneworld black hole and analyze its perturbations. We develop the perturbation formalism for Randall-Sundrum model from first principals and discuss the weak field limit of the model in the solar system. We derive explicit equations of motion for the axial and spherical gravitational waves in the black string background. These are solved numerically in various scenarios, and the characteristic late-time signal from a black string is obtained. We find that if one waits long enough after some transient event, the signal from the string will be a superposition of nearly monochromatic waves with frequencies corresponding to the masses of the Kaluza Klein modes of the model. We estimate the amplitude of the spherical component of these modes when they are excited by a point particle orbiting the string.

1 Introduction

Braneworld models hypothesize that our observable universe is a hypersurface, called the 'brane', embedded in some higher-dimensional spacetime. Standard model particles and fields are assumed to be confined to the brane, while gravitational degrees of freedom are free to propagate in the full higher-dimensional 'bulk'. The phenomenological implications of these models have been intensively studied by many different authors over the past decade, with great emphasis being placed on any observational consequences of the existence of large, possibly infinite, extra dimensions.

There are a number of different braneworld models, but perhaps one of the best studied is the Randall-Sundrum (RS) scenario (15; 16). There are two variants of the model involving either one or two branes, but the common assumption in both setups is that there is a negative cosmological constant in the bulk characterized by

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a curvature scale ℓ. The great virtue of the model is that the gravity behaves like ordinary general relativity (GR) in 'weak field' situations; i.e., when the density of matter is small or scale of interest is large. In particular, one recovers the Newtonian inverse-square law of gravitation in the RS model as long as the separation between the two bodies $\gg \ell$. This leads to a direct laboratory constraint on the bulk curvature scale, since Newton's law is known to be valid on scales larger than around $50\,\mu$m (10).

The RS model is also consistent with various astrophysical tests of GR in the weak field regime, including the solar system tests such as the perihelion shift of Mercury or time delay experiments using the Cassini spacecraft. On the cosmological side, one can also demonstrate that the RS predictions for the dynamics of the scale factor or the growth of fluctuations match the predictions of GR as long as the Hubble horizon H^{-1} is less that the AdS length scale ℓ. Hence, the RS model matches conventional theory in the low-energy universe.

The ability of the RS model to mimic GR in these cases is both fortuitous and somewhat surprising. The introduction of a large extra dimension is not a trivial modification of standard theory, and before the work of Randall & Sundrum the conventional wisdom was that such models could not be made to be consistent with the real measured behaviour of gravity. The fact that a fifth dimension can be made to conform to what we observe is part of the reason for the flurry of activity on the RS model since its inception. It also raises an interesting problem: The correspondence between GR and the RS scenario must fail at some point, since at the end of the day they have very different geometric setups. In what situations does this breakdown occur, and are there any associated observational signatures that we can use to constrain the RS model?

We mentioned above that RS cosmology matches GR cosmology for $H\ell \lesssim 1$. Thus, we are led to look for deviations from standard theory in cosmological epochs with $H\ell \gtrsim 1$. This corresponds to the very high-energy radiation epoch, which is just after inflation and before nucleosynthesis. People have looked at modifications to the background expansion, dynamics of gravitational waves (9; 11; 17), and the growth of density perturbations in the high-energy epoch (1). All of these phenomena show some departures from GR, but as of yet there has been no clean observational test proposed that could either rule out or rule in the RS model.

Hence, we need to look to other 'strong field' scenarios to test the model. One possibility is to look at black holes in the Randall-Sundrum model. We know that these objects are not describable in the Newtonian limit of GR, so one might expect that braneworld black holes to exhibit observable deviations from the ordinary Schwarzschild or Kerr solutions. However, there is a major problem with using black holes a probe of braneworld models: There is no known 'reasonable' brane-localized black hole solution in the RS one brane scenario. The lack of a solution is not for lack of trying, many authors have attempted various techniques to find one. One of the first attempts was using the 5-dimensional black string solution as a bulk manifold (2). However, it was demonstrated that such solutions were subject to the famous Gregory-Laflamme instability (8), which is a tachyonic mode with a long wavelength in the extra dimension. Others have tried to find brane black holes nu-

merically (14), but success has been limited to small mass objects $GM \ll \ell$. Several have conjectured that the lack of a solution in the one brane case has to do with the AdS/CFT correspondence (5; 19).

However, the situation is somewhat better in the two brane case. It turns out that it is possible to find a stable braneworld model in this case, and that the brane geometry is exactly 4-dimensional Schwarzschild (3; 4; 18). Like the model considered in (2), this is based on the 5-dimensional black string. The Gregory-Laflamme instability is evaded by the infrared cutoff introduced by the second brane; i.e., the model is stable if the branes are close enough together. Because the geometry on the brane is identical to that of the Schwarzschild metric, the model is automatically in agreement with any test of GR sensitive to the background geometry only; such as light-bending, perihelion shifts, time delays, etc.

Hence, we need to look at the perturbative aspects of the model to obtain differences with ordinary GR. In particular, we are interested in the gravitational waves (GWs) emitted from these black strings when they are displaced from their equilibrium configuration. Of primary importance is the issue of whether or not any deviations from the predictions of GR are observable by GW detectors such as LIGO or LISA. These issues are the subject of these lecture notes.

In §2 we introduce the RS model and the black string braneworld. In §3, we describe how to perturb the model and derive the relevant equations of motion. In §4, we show how to separate variables in the governing partial differential equations (PDEs) by introducing the Kaluza-Klein (KK) decomposition. In §5, we consider the limit under which we recover GR. In §6, we define the complete mode decomposition in terms of KK modes and spherical harmonics used in the rest of the notes. In §7, we consider homogeneous solutions to the axial equations of motion and determine (via simulations) the characteristic GW signal produced by the string. In §8, we consider the spherical sector of the GW spectrum excited by generic sources and discuss the Gregory-Laflamme instability in detail. In §9, we write down explicit equations of motion for the spherical GWs emitted by a point particle orbiting the black string and consider their numeric solution. In §10, we estimate the amplitude of Kaluza-Klein radiation emitted from the black string for a given point particle source. Finally, in §11 we give a brief summary and outline some open questions.

2 A generalized Randall-Sundrum two brane model

In this section, we present a generalized version of the Randall-Sundrum two brane model in a coordinate invariant formalism. We begin by outlining the geometry of the model, the action governing the dynamics, and the ensuing field equations. We then specialize to the black string braneworld model, which will be perturbed in the next section.

2.1 Geometrical framework and notation

Consider a (4+1)-dimensional manifold $(\mathscr{M},g)$, which we refer to as the 'bulk'. One of the spatial dimensions of $\mathscr{M}$ is assumed to be compact; i.e., the 5-dimensional topology is $\mathbb{R}^4\times S$. We place coordinates x^A on $\mathscr{M}$ so that the 5-dimensional line element reads:

$$ds_5^2 = g_{AB}dx^A dx^B. \tag{1}$$

We assume that there is a scalar function Φ that uniquely maps points in $\mathscr{M}$ into the interval $I=(-d,+d]$. Here, d is a constant parameter that is one of the fundamental length scales of the problem. The gradient of this mapping $\partial_A\Phi$ is spacelike,

$$\partial_A\Phi\,\partial^A\Phi > 0, \tag{2}$$

and is tangent to the compact dimension of $\mathscr{M}$. This scalar function defines a family of timelike hypersurfaces $\Phi(x^A)=Y$, which we denote by Σ_Y. The two submanifolds at the endpoints of I, Σ_d and Σ_{-d}, are periodically identified.

Let us now place 4-dimensional coordinates z^α on each of the Σ_Y hypersurfaces. These coordinates will be related to their 5-dimensional counterparts by parametric equations of the form: $x^A = x^A(z^\alpha)$. We then define the following basis vectors

$$e^A_\alpha = \frac{\partial x^A}{\partial z^\alpha}, \quad n^A = \frac{\partial^A\Phi}{\sqrt{\partial_B\Phi\,\partial^B\Phi}}, \qquad n_A e^A_\alpha = 0, \quad n^A n_A = +1. \tag{3}$$

The tetrad e^A_α is everywhere tangent to Σ_Y, while n^A is everywhere normal to Σ_Y. The projection tensor onto the Σ_Y hypersurfaces is given by

$$q_{AB} = g_{AB} - n_A n_B, \quad n^A q_{AB} = 0. \tag{4}$$

From this, it follows that the intrinsic line element on each of the Σ_Y hypersurfaces is

$$ds_4^2 = q_{\alpha\beta}dz^\alpha dz^\beta, \quad q_{\alpha\beta} = e^A_\alpha e^B_\beta q_{AB} = e^A_\alpha e^B_\beta g_{AB}. \tag{5}$$

The object $q_{\alpha\beta}$ behaves as a tensor under 4-dimensional coordinate transformations $z^\alpha \to \tilde{z}^\alpha(z^\beta)$ and is the induced metric on the Σ_Y hypersurfaces. It has an inverse $q^{\alpha\beta}$ that can be used to define e^α_A:

$$e^\alpha_A = g_{AB}q^{\alpha\beta}e^B_\beta, \quad \delta^\alpha_\beta = q^{\alpha\gamma}q_{\gamma\beta} = e^\alpha_A e^A_\beta. \tag{6}$$

Generally speaking, we define the projection of any 5-tensor T_{AB} onto the Σ_Y hypersurfaces as

$$T_{\alpha\beta} = e^A_\alpha e^B_\beta T_{AB}, \tag{7}$$

where the generalization to tensors of other ranks is obvious. The 4-dimensional intrinsic covariant derivative of $T_{\alpha\beta}$ is related to the 5-dimensional covariant derivative of T_{AB} by

$$[\nabla_\alpha T_{\mu\nu}]_q = e^A_\alpha e^M_\mu e^N_\nu \nabla_A q^B_M q^C_N T_{BC}, \tag{8}$$

where the notation $[\cdots]_q$ means that the quantity inside the square brackets is calculated with the $q_{\alpha\beta}$ metric.

Finally, the extrinsic curvature of each Σ_Y hypersurface is:

$$K_{AB} = q_A^C \nabla_C n_B = \tfrac{1}{2}\pounds_n q_{AB} = K_{BA}, \quad n^A K_{AB} = 0,$$
$$K_{\alpha\beta} = e_\alpha^A e_\beta^B K_{AB} = e_\alpha^A e_\beta^B \nabla_A n_B. \tag{9}$$

2.2 The action and field equations

We label the hypersurfaces at $Y = y_+ = 0$ and $Y = y_- = +d$ as the 'visible brane' Σ^+ and 'shadow brane' Σ^-, respectively. Our observable universe is supposed to reside on the visible brane. These hypersurfaces divide the bulk into two halves: the lefthand portion $\mathscr{M}_\mathrm{L}$ which has $y \in (-d, 0)$, and the righthand portion which has $y \in (0, +d)$. The action for our model is:

$$S = \frac{1}{2\kappa_5^2}\int\limits_{\mathscr{M}_\mathrm{L}} \left[{}^{(5)}R - 2\Lambda_5\right] + \frac{1}{2\kappa_5^2}\int\limits_{\mathscr{M}_\mathrm{R}} \left[{}^{(5)}R - 2\Lambda_5\right]$$
$$+ \sum_{\varepsilon=\pm}\frac{1}{2}\int\limits_{\Sigma^\varepsilon}\left(\mathscr{L}^\varepsilon - 2\lambda^\varepsilon - \frac{1}{\kappa_5^2}[K]^\varepsilon\right) + \frac{1}{2}\int\limits_{\mathscr{M}_\mathrm{L}}\mathscr{L}_\mathrm{L} + \frac{1}{2}\int\limits_{\mathscr{M}_\mathrm{R}}\mathscr{L}_\mathrm{R}. \tag{10}$$

In this expression, κ_5^2 is the 5-dimensional gravity matter coupling, $\Lambda_5 = -6k^2$ is the bulk cosmological constant, $\lambda^\pm = \pm 6k/\kappa_5^2$ are the brane tensions, and $\ell = 1/k$ is the curvature length scale of the bulk. Also, $\mathscr{L}^\pm$ is the Lagrangian density of matter residing on $\Sigma^\pm$, while $\mathscr{L}_\mathrm{L}$ and $\mathscr{L}_\mathrm{R}$ are the Lagrangian densities of matter living in the bulk. Note that the visible brane in our model has positive tension while the shadow brane has negative tension.

The quantity $[K]^\pm$ is the jump in the trace of the extrinsic curvature of the Σ_Y hypersurfaces across each brane. To clarify, suppose that $\partial\mathscr{M}_\mathrm{L}^\pm$ and $\partial\mathscr{M}_\mathrm{R}^\pm$ are the boundaries of $\mathscr{M}_L$ and $\mathscr{M}_R$ coinciding with $\Sigma^\pm$, respectively. Then,

$$[K]^+ = q^{\alpha\beta}K_{\alpha\beta}\Big|_{\partial\mathscr{M}_\mathrm{R}^+} - q^{\alpha\beta}K_{\alpha\beta}\Big|_{\partial\mathscr{M}_\mathrm{L}^+}, \tag{11a}$$
$$[K]^- = q^{\alpha\beta}K_{\alpha\beta}\Big|_{\partial\mathscr{M}_\mathrm{L}^-} - q^{\alpha\beta}K_{\alpha\beta}\Big|_{\partial\mathscr{M}_\mathrm{R}^-}. \tag{11b}$$

We can now write down the field equations for our model. Setting the variation of S with respect to the bulk metric g^{AB} equal to zero yields that:

$$G_{AB} - 6k^2 g_{AB} = \kappa_5^2\left[\theta(+y)T_{AB}^\mathrm{R} + \theta(-y)T_{AB}^\mathrm{L}\right],$$
$$T_{AB}^{\mathrm{L,R}} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathscr{L}_{\mathrm{L,R}}\right)}{\delta g^{AB}}. \tag{12}$$

Meanwhile, variation of S with respect to the induced metric on each boundary yields

$$Q^{\pm}_{AB} = \left\{ [K_{AB}] \pm 2kq_{AB} + \kappa_5^2 (T_{AB} - \tfrac{1}{3} T q_{AB}) \right\}^{\pm} = 0, \tag{13a}$$

$$T^{\pm}_{AB} = e^{\alpha}_{A} e^{\beta}_{B} \left\{ -\frac{2}{\sqrt{-q}} \frac{\delta\left(\sqrt{-q}\mathscr{L}\right)}{\delta q^{\alpha\beta}} \right\}^{\pm}. \tag{13b}$$

Here, the $\{\cdots\}^{\pm}$ notation means that everything inside the curly brackets is evaluated at $\Sigma^{\pm}$. We see that (12) are the bulk field equations to be satisfied by the 5-dimensional metric g_{AB}, while (13) are the boundary conditions that must be enforced at the position of each brane. Of course, (13) are simply the Israel junction conditions for thin shells in general relativity.

In what sense is our model a generalization of the RS setup? The original Randall-Sundrum model exhibited a $\mathbb{Z}_2$ symmetry, which implied that $\mathscr{M}_{\mathrm{L}}$ is the mirror image of $\mathscr{M}_{\mathrm{R}}$. Also, in the RS model the bulk was explicitly empty. However, since we allow for an asymmetric distribution of matter in the bulk, we explicitly violate the $\mathbb{Z}_2$ symmetry and bulk vacuum assumption.

2.3 The black string braneworld

We now introduce the black string braneworld, which is a $\mathbb{Z}_2$ symmetric solution of (12) and (13) with no matter sources:

$$\mathscr{L}_{\mathrm{L}} \doteq \mathscr{L}_{\mathrm{R}} \doteq \mathscr{L}^{\pm} \doteq 0. \tag{14}$$

Here, we use $\doteq$ to indicate equalities that only hold in the black string background. The bulk geometry for this solution is given by:

$$ds_5^2 \doteq a^2(y) \left[-f(r)\, dt^2 + \frac{1}{f(r)}\, dr^2 + r^2\, d\Omega^2 \right] + dy^2, \tag{15a}$$

$$f(r) = 1 - 2GM/r, \quad a(y) = \mathrm{e}^{-k|y|}. \tag{15b}$$

Here, M is the mass parameter of the black string and $G = \ell_{\mathrm{Pl}}/M_{\mathrm{Pl}}$ is the ordinary 4-dimensional Newton's constant. The function Φ used to locate the branes is trivial in this background:

$$\Phi(x^A) \doteq y, \tag{16}$$

which means that the $\Sigma^{\pm}$ branes are located at $y = 0$ and $y = d$, respectively. The $\Sigma_Y \doteq \Sigma_y$ hypersurfaces have the geometry of Schwarzschild black holes, and there is 5-dimensional line-like curvature singularity at $r = 0$:

$$R^{ABCD} R_{ABCD} \doteq \frac{48 G^2 M^2 e^{4k|y|}}{r^6} + 40k^2. \tag{17}$$

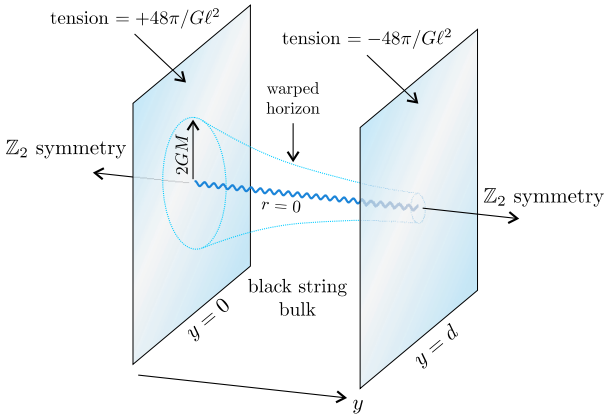

Fig. 1 A schematic illustration of the black string braneworld

Note that the other singularities at $y = \pm\infty$ are excised from our model by the restriction $y \doteq Y \in (-d, d]$, so we will not consider them further. An illustration of the black string braneworld background is given in Figure 1.

We remark that it is actually possible to replace the 4-metric in square brackets in (15) by any 4-dimensional solution of $R_{\alpha\beta} = 0$ and still satisfy the 5-dimensional field equation. That is, we could have

$$ds_5^2 \doteq a(y)^2 ds_{\text{Kerr}}^2 + dy^2, \tag{18}$$

where ds_{Kerr}^2 is the line element corresponding to the Kerr solution for a rotating black hole. Such a solution is known as the rotating black string. The dynamics of perturbations of the rotating black string are still an open question due to the extreme complexity of the governing equations of motion.

Finally, note that the normal and extrinsic curvature associated with the Σ_Y hypersurfaces satisfy the following convenient properties:

$$n_A \doteq \partial_A y, \quad n^A \nabla_A n^B \doteq 0, \quad K_{AB} \doteq -k q_{AB}. \tag{19}$$

These expressions are used liberally below to simplify formulae evaluated in the black string background.

3 Linear perturbations

We now turn our attention to perturbations of the black sting braneworld. We first describe the perturbative variable we use to describe the fluctuations of the system, then we linearize the bulk field equations and junction conditions. We finish this section by rewriting the perturbative equations of motion in a particularly useful

form. Note that while we work from first principals in §3–§5, similar calculations and results have appeared many times in the literature; see the seminal works by Randall & Sundrum (15) and Garriga & Tanaka (6), for example.

3.1 Perturbative variables

We are ultimately interested in the behaviour of gravitational waves in this model, which are described by fluctuations of the bulk metric:

$$g_{AB} \to g_{AB} + h_{AB}, \tag{20}$$

where h_{AB} is understood to be a 'small' quantity. The projection of h_{AB} onto the visible brane is the observable that can potentially be measured in gravitational wave detectors. But it is not sufficient to consider fluctuations in the bulk metric alone — to get a complete picture, we must also allow for the perturbation of the matter content of the model as well as the positions of the branes.

Obviously, matter perturbations are simply described by the T^{L}_{AB}, T^{R}_{AB}, and $T^{\pm}_{AB}$ stress-energy tensors, which are considered to be small quantities of the same order as h_{AB}. On the other hand, we describe fluctuations in the brane positions via a perturbation of the scalar function Φ:

$$\Phi(x^A) \to y + \xi(x^A). \tag{21}$$

Here, ξ is a small spacetime scalar. Recall that the position of each brane is implicitly defined by $\Phi(x^A) = y_\pm$. Hence, the brane locations after perturbation are given by the solution of the following for y:

$$y + \xi\Big|_{y=y_\pm} + (y - y_\pm)\partial_y \xi\Big|_{y=y_\pm} + \cdots = y_\pm. \tag{22}$$

However, note that $y - y_\pm$ is of the same order as ξ, so at the linear level the new brane positions are simply given by

$$y = y_\pm - \xi\Big|_{y=y_\pm}. \tag{23}$$

Hence, the perturbed brane positions are given by the brane bending scalars:

$$\xi^\pm = \xi\Big|_{y=y_\pm}, \quad n^A \partial_A \xi^\pm = 0. \tag{24}$$

Note that because ξ^+ and ξ^- are explicitly evaluated at the brane positions, they are essentially 4-dimensional scalars that exhibit no dependence on the extra dimension.

Having now delineated a set of variables that parameterize the fluctuations of the black string braneworld, we now need to determine their equations of motion.

3.2 Linearizing the bulk field equations

First, we linearize the bulk field equations (12) about the black string solution. Notice that (12) only depends on the bulk metric and the bulk matter distribution. Hence, the linearized field equations will only involve h_{AB}, $T^{\rm L}_{AB}$ and $T^{\rm R}_{AB}$. The actual derivation of the equation proceeds in the same manner as in 4-dimensions, and we just quote the result:

$$\nabla^C\nabla_C h_{AB} - \nabla^C\nabla_A h_{BC} - \nabla^C\nabla_B h_{AC} + \nabla_A\nabla_B h^C{}_C - 8k^2 h_{AB} = -2\kappa_5^2\Sigma^{\rm bulk}_{AB}, \qquad (25)$$

where

$$\Sigma^{\rm bulk}_{AB} = \Theta(+y)(T^{\rm R}_{AB} - \tfrac{1}{3}T^{\rm R}g_{AB}) + \Theta(-y)(T^{\rm L}_{AB} - \tfrac{1}{3}T^{\rm L}g_{AB}). \qquad (26)$$

The wave equation (25) is valid for arbitrary choices of gauge and generic matter sources. If we specialize to the Randall-Sundrum gauge

$$\nabla^A h_{AB} = 0, \quad h^A{}_A = 0, \quad h_{AB} = e^\alpha_A e^\beta_B h_{\alpha\beta}, \qquad (27)$$

eq. (25) reduces to

$$\hat{\Delta}_{AB}{}^{CD} h_{CD} + (GMa)^2(\pounds_n^2 - 4k^2)h_{AB} = -2(GMa)^2\kappa_5^2\Sigma^{\rm bulk}_{AB}, \qquad (28)$$

where we have defined the operator

$$\begin{aligned}\hat{\Delta}_{AB}{}^{CD} =& (GMa)^2[q^{MN}\nabla_M q^P_N q^C_A q^D_B \nabla_P + 2^{(4)}R_A{}^C{}_B{}^D] \\ =& (GMa)^2 e^\alpha_A e^\beta_B \Big[\delta^\gamma_\alpha\delta^\delta_\beta\nabla^\rho\nabla_\rho + 2R_\alpha{}^\gamma{}_\beta{}^\delta\Big]_q e^C_\gamma e^D_\delta \\ =& (GM)^2 e^\alpha_A e^\beta_B \Big[\delta^\gamma_\alpha\delta^\delta_\beta\nabla^\rho\nabla_\rho + 2R_\alpha{}^\gamma{}_\beta{}^\delta\Big]_g e^C_\gamma e^D_\delta. \end{aligned} \qquad (29)$$

Here, ${}^{(4)}R_{ACBD}$ is the Riemann tensor on Σ_y, which can be related to the 5-dimensional curvature tensor via the Gauss equation

$${}^{(4)}R_{MNPQ} = q^A_M q^B_N q^C_P q^D_Q R_{ABCD} + 2K_{M[P}K_{Q]N}. \qquad (30)$$

On the second line of (29) the 4-tensor inside the square brackets is calculated using $q_{\alpha\beta}$. We can re-express this object in terms of the ordinary Schwarzschild metric $g_{\alpha\beta}$, which is conformally related to $q_{\alpha\beta}$ via the warp factor:

$$q_{\alpha\beta} = a^2 g_{\alpha\beta}, \qquad (31a)$$

$$g_{\alpha\beta}dz^\alpha dz^\beta = -f\,dt^2 + f^{-1}dr^2 + r^2 d\Omega^2. \qquad (31b)$$

The quantity in square brackets on the third line of (29) is calculated from $g_{\alpha\beta}$.[1] One can easily confirm that $\hat{\Delta}_{AB}{}^{CD}$ is 'y-independent' in the sense that it commutes

[1] Unless otherwise indicated, for the rest of the paper any tensorial expression with Greek indices should be evaluated using the Schwarzschild metric $g_{\alpha\beta}$.

with the Lie derivative in the n^A direction:

$$[{}^{(4)}\hat{\Delta}_{AB}{}^{CD}, \pounds_n] = 0. \tag{32}$$

In addition, the $(GM)^2$ prefactor makes $\hat{\Delta}_{AB}{}^{CD}$ dimensionless.

Notice that the lefthand side of (28) is both traceless and manifestly orthogonal to n^A, which implies the following constraints on the bulk matter:

$$\Sigma^{\text{bulk}}_{AB} = e^\alpha_A e^\beta_B \Sigma^{\text{bulk}}_{\alpha\beta}, \quad q^{\alpha\beta}\Sigma^{\text{bulk}}_{\alpha\beta} = 0. \tag{33}$$

In other words, our gauge choice is inconsistent with bulk matter that violates these conditions. If we wish to consider more general bulk matter, we cannot use the Randall-Sundrum gauge.

3.3 Linearizing the junction conditions

Next, we consider the perturbation of the junction conditions (13). These can be re-written as

$$Q^\pm_{AB} = \left\{ [\tfrac{1}{2}\nabla_{(A} n_{B)} - n_{(A|} n^C \nabla_C n_{|B)}] \pm k q_{AB} + \kappa_5^2 \left(T_{AB} - \tfrac{1}{3} T q_{AB} \right) \right\}^\pm = 0. \tag{34}$$

We require that $Q^\pm_{AB}$ vanish before and after perturbation, so we need to enforce that the first order variation $\delta Q^\pm_{AB}$ is equal to zero.

In order to calculate this variation, we can regard the tensors $Q^\pm_{AB}$ as functionals the brane positions (as defined by Φ), the brane normals n_A, the bulk metric, and the brane matter:

$$Q^\pm_{AB} = Q^\pm_{AB}(\Phi, n_M, g_{MN}, T^\pm_{MN}), \tag{35}$$

from which it follows that

$$\delta Q^\pm_{AB} = \left\{ \frac{\delta Q_{AB}}{\delta \Phi}\delta\Phi + \frac{\delta Q_{AB}}{\delta n_C}\delta n_C + \frac{\delta Q_{AB}}{\delta g_{CD}}\delta g_{CD} + \frac{\delta Q_{AB}}{\delta T_{CD}}\delta T_{CD} \right\}^\pm_0. \tag{36}$$

The $\{\cdots\}^\pm_0$ notation is meant to remind us that after we have calculated the variational derivatives, we must evaluate the expression in the background geometry at the *unperturbed* positions of the brane.

We now consider each term in (36). For simplicity, we temporarily focus on the positive tension visible brane and drop the + superscript. The first term represents the variation of $Q^\pm_{AB}$ with brane position, which is covariantly given by the Lie derivative in the normal direction:

$$\left\{ \frac{\delta Q_{AB}}{\delta \Phi}\delta\Phi \right\}_0 = \{ -\xi \pounds_n Q_{AB} \}_0 . \tag{37}$$

But the Lie derivative of Q_{AB} vanishes identically in the background geometry, so this term is equal to zero.

The second term in (36) represents the variation of Q_{AB} with respect to the normal vector. Making note of the definition (3) of n^A in terms of Φ, as well as $\delta\Phi = \xi$ and $n^A\nabla_A\xi = 0$, we arrive at

$$\delta n_A = \nabla_A\xi, \quad n^A\delta n_A = 0. \tag{38}$$

Notice that since the normal itself must be continuous across the brane, we have $[\delta n_A] = 0$. After some algebra, we find that the variation of the junction conditions with respect to the brane normal is non-zero and given by

$$\left\{\frac{\delta Q_{AB}}{\delta n_C}\delta n_C\right\}_0 = 2q_A^C q_B^D \nabla_C\nabla_D\xi. \tag{39}$$

The third term in (36) is the variation with the bulk metric itself $\delta g_{AB} = h_{AB}$. Calculating this is straightforward, and the result is:

$$\left\{\frac{\delta Q_{AB}}{\delta g_{CD}}\delta g_{CD}\right\}_0 = \tfrac{1}{2}[\pounds_n h_{AB}] + 2kh_{AB}. \tag{40}$$

The last variation we must consider is with respect to the brane matter fields, which is trivial:

$$\left\{\frac{\delta Q_{AB}}{\delta T_{CD}}\delta T_{CD}\right\}_0 = \kappa_5^2\left(T_{AB} - \tfrac{1}{3}Tq_{AB}\right). \tag{41}$$

So, we have the final result that

$$\delta Q_{AB}^{\pm} = \left\{2q_A^C q_B^D\nabla_C\nabla_D\xi + \tfrac{1}{2}[\pounds_n h_{AB}] \pm 2kh_{AB} + \kappa_5^2\left(T_{AB} - \tfrac{1}{3}Tq_{AB}\right)\right\}_0^{\pm} = 0. \tag{42}$$

If we take the trace of $\delta Q_{AB}^{\pm} = 0$, we obtain

$$q^{AB}\nabla_A\nabla_B\xi^{\pm} = \tfrac{1}{6}\kappa_5^2 T^{\pm}. \tag{43}$$

These are the equations of motion for the brane bending degrees of freedom in our model, which are seen to be directly sourced by the matter fields on each brane.

3.4 Converting the boundary conditions into distributional sources

We can incorporate the boundary conditions $\delta Q_{AB}^{\pm} = 0$ directly into the h_{AB} equation of motion as delta-function sources. This is possible because the jump in the normal derivative of h_{AB} appears explicitly in the perturbed junction conditions. This procedure gives

$$\hat{\Delta}_{AB}{}^{CD} h_{CD} - \hat{\mu}^2 h_{AB} = -2(GMa)^2 \kappa_5^2 \left[\Sigma_{AB}^{\text{bulk}} + \sum_{\varepsilon=\pm} \delta(y-y_\varepsilon) \Sigma_{AB}^{\varepsilon} \right]. \tag{44}$$

Here, we have defined

$$\begin{aligned} \hat{\mu}^2 &= -(GMa)^2 \left[\pounds_n^2 + \frac{2\kappa_5^2}{3} \sum_{\varepsilon=\pm} \lambda^\varepsilon \delta(y-y_\varepsilon) - 4k^2 \right], \\ \Sigma_{AB}^{\pm} &= \left(T_{AB}^{\pm} - \tfrac{1}{3} T^{\pm} q_{AB} \right) + \frac{2}{\kappa_5^2} q_A^C q_B^D \nabla_C \nabla_D \xi^{\pm}. \end{aligned} \tag{45}$$

If we integrate the wave equation (44) over a small region traversing either brane, we recover the boundary conditions (42).

Together with the gauge conditions,

$$n^A h_{AB} = q^{AC} \nabla_A h_{CB} = 0 = q^{AB} h_{AB}, \tag{46}$$

(43) and (44) are the equations governing the perturbations of our model.

4 Kaluza-Klein mode functions

The metric fluctuation h_{AB} is governed by a system of partial differential equations (PDEs). As is common in all areas of physics, the best way to solve such equations is via a separation of variables. In this section, we separate the y variables from the conventional Schwarzschild variables on Σ_y. The part of the graviton wave function corresponding to the extra dimension satisfies an ODE boundary value problem, which implies that there is a discrete spectrum for h_{AB}.

4.1 Separation of variables

As mentioned above, we have that

$$[\hat{\Delta}_{AB}{}^{CD}, \pounds_n] h_{CD} = 0; \tag{47}$$

i.e., $\hat{\Delta}_{AB}{}^{CD}$ is independent of y when evaluated in the (t, r, θ, ϕ, y) coordinates. This suggests that we seek a solution for h_{AB} of the form

$$h_{AB} = Z \tilde{h}_{AB}, \quad \hat{\mu}^2 Z = \mu^2 Z, \tag{48}$$

where,

$$0 = \pounds_n \tilde{h}_{AB} \text{ and } 0 = q_B^A \nabla_A Z; \tag{49}$$

that is, Z is an eigenfunction of $\hat{\mu}^2$ with eigenvalue μ^2. The existence of the delta functions in the $\hat{\mu}^2$ operator means that we need to treat the even and odd parity solutions of this eigenvalue problem separately.

4.2 Even parity eigenfunctions

If $Z(-y) = Z(y)$, we see that Z satisfies the following equations in the interval $y \in [0,d]$:

$$\begin{aligned} m^2 Z(y) &= -a^2(y)(\partial_y^2 - 4k^2)Z(y), \\ 0 &= [(\partial_y + 2k)Z(y)]_{\pm}, \\ \mu &= GMm. \end{aligned} \tag{50}$$

There is a discrete spectrum of solutions to this eigenvalue problem that are labeled by the positive integers $n = 1,2,3\ldots$:

$$Z_n(y) = \alpha_n^{-1}[Y_1(m_n\ell)J_2(m_n\ell e^{k|y|}) - J_1(m_n\ell)Y_2(m_n\ell e^{k|y|})], \tag{51}$$

where α_n is a constant, and $m_n = \mu_n/GM$ is the n^{th} solution of

$$Y_1(m_n\ell)J_1(m_n\ell e^{kd}) = J_1(m_n\ell)Y_1(m_n\ell e^{kd}). \tag{52}$$

There is also a solution corresponding to $m_0 = \mu_0 = 0$, which is known as the zero-mode:

$$Z_0(y) = \alpha_0^{-1} e^{-2k|y|}, \quad \alpha_0 = \sqrt{\ell}(1 - e^{-2kd})^{1/2}. \tag{53}$$

Hence, there exists a discrete set of solutions for bulk metric perturbations of the form $h_{AB}^{(n)} = Z_n(y)\tilde{h}_{AB}^{(n)}(z^\alpha)$. When $n > 0$ these are called the Kaluza-Klein (KK) modes of the modes, and the mass of any given mode is given by the m_n eigenvalue. The α_n constants are determined from demanding that $\{Z_n\}$ forms an orthonormal set

$$\delta_{mn} = \int_{-d}^{d} dy\, a^{-2}(y) Z_m(y) Z_n(y). \tag{54}$$

These basis functions then satisfy:

$$\delta(y - y_\pm) = \sum_{n=0}^{\infty} a^{-2} Z_n(y) Z_n(y_\pm). \tag{55}$$

This identity is crucial to the model — inspection of (44) reveals that the brane stress energy tensors appearing on the righthand side are multiplied by one of $\delta(y - y_\pm)$. Hence, brane matter only couples to the even parity eigenmodes of $\hat{\mu}^2$.

Case 1: light modes

It is useful to have simple approximate forms of the Kaluza-Klein masses and normalization constants for the formulae that appear later on. There are straightforward to derive for modes that are 'light' compared to mass scale set by the AdS_5 length parameter:

$$m_n \ell \ll 1. \tag{56}$$

Let us define a set of dimensionless numbers x_n by:

$$x_n = m_n \ell e^{kd}. \tag{57}$$

Then for the light modes, we find that x_n is the n^{th} zero of the first-order Bessel function:

$$J_1(x_n) = 0. \tag{58}$$

Also for light modes, the normalization constants reduce to

$$\alpha_n \approx 2\sqrt{\ell}\, e^{2kd} |J_0(x_n)| / \pi x_n, \quad n > 0. \tag{59}$$

Actually, it is more helpful to know the value of the KK mode functions at the position of each brane. We can parameterize these as

$$Z_n(y_\pm) = \sqrt{k} e^{-kd} z_n^\pm, \quad n > 0. \tag{60}$$

For the light Kaluza-Klein modes, the dimensionless $z_n^\pm$ are given by

$$z_n^\pm \approx \left\{ \begin{array}{c} |J_0(x_n)|^{-1} \\ e^{in\pi} \end{array} \right\}. \tag{61}$$

Case 2: heavy modes

At the other end of the spectrum, we have the heavy Kaluza-Klein modes

$$m_n \ell \gg 1. \tag{62}$$

Under this assumption, we find[2]

[2] Strictly speaking, an asymptotic analysis leads to formulae with n replaced by another integer n' on the righthand sides of Eqns. (63). However, we note that for even parity modes, n counts the number of zeroes of $Z_n(y)$ in the interval $y \in (0, d)$, which allows us to deduce that $n' = n$.

$$x_n \approx \frac{n\pi}{1-e^{-kd}}, \tag{63a}$$

$$Z_n(y) \approx \sqrt{\frac{ke^{-k|y|}}{e^{kd}-1}}\cos\left[n\pi\frac{e^{k|y|}-1}{e^{kd}-1}\right], \tag{63b}$$

$$z_n^{\pm} \approx \frac{1}{\sqrt{1-e^{-kd}}}\left\{\begin{array}{c} e^{kd/2} \\ e^{in\pi}\end{array}\right\}. \tag{63c}$$

Unlike the analogous quantities for the light modes, $z_n^{\pm}$ shows an explicit dependence on the dimensionless brane separation d/ℓ.

4.3 Odd parity eigenfunctions

As mentioned above, brane matter only couples to Kaluza-Klein modes with even parity. But a complete perturbative description must include the odd parity modes as well; for example, if we have matter in the bulk distributed asymmetrically with respect to $y=0$ (i.e. $T^{\mathrm{L}}_{AB} \neq T^{\mathrm{R}}_{AB}$) modes of either parity will be excited. Hence, for the sake of completeness, we list a few properties of the odd parity Kaluza-Klein modes here.

Assuming $Z(-y) = -Z(y)$, we have:

$$\begin{aligned} m^2 Z(y) &= -a^2(y)(\partial_y^2 - 4k^2)Z(y), \\ 0 &= Z(y_+) = Z(y_-). \end{aligned} \tag{64}$$

Again, we have a discrete spectrum of solutions, this time labeled by half integers:

$$Z_{n+\frac{1}{2}}(y) = \alpha^{-1}_{n+\frac{1}{2}}\left[Y_2(m_{n+\frac{1}{2}}\ell)J_2(m_{n+\frac{1}{2}}\ell e^{k|y|}) - J_2(m_{n+\frac{1}{2}}\ell)Y_2(m_{n+\frac{1}{2}}\ell e^{k|y|})\right]. \tag{65}$$

The mass eigenvalues are now the solutions of

$$Y_2(m_{n+\frac{1}{2}}\ell)J_2(m_{n+\frac{1}{2}}\ell e^{kd}) = J_2(m_{n+\frac{1}{2}}\ell)Y_2(m_{n+\frac{1}{2}}\ell e^{kd}). \tag{66}$$

Proceeding as before, we define

$$x_{n+\frac{1}{2}} = m_{n+\frac{1}{2}}\ell e^{kd}. \tag{67}$$

For light modes with $m_{n+\frac{1}{2}}\ell \ll 1$, $x_{n+\frac{1}{2}}$ is the n^{th} zero of the second-order Bessel function:

$$J_2(x_{n+\frac{1}{2}}) = 0. \tag{68}$$

Taken together, (58) and (68) imply the following for the light modes:

$$m_1 < m_{3/2} < m_2 < m_{5/2} < \cdots; \tag{69}$$

i.e., the first odd mode is heavier than the first even mode, etc.

Finally, we note that since the odd modes vanish at the background position of the visible brane, it is impossible for us to observe them directly within the context of linear theory. This can change at second order, since brane bending can allow us to directly sample regions of the bulk where $Z_{n+\frac{1}{2}} \neq 0$. However, this phenomenon is clearly beyond the scope of this paper.

5 Recovering 4-dimensional gravity

Let us now describe the limit in which we recover general relativity. We assume there are no matter perturbations in the bulk and on the hidden brane; hence, we may consistently neglect the odd parity Kaluza-Klein modes. By virtue of the brane bending equation of motion (43), we can consistently set $\xi^- = 0$. Furthermore, (55) can be used to replace the delta function in front of Σ^+_{AB} in equation (44). We obtain,

$$\hat{\Delta}_{AB}{}^{CD} h_{CD} - \hat{\mu}^2 h_{AB} = -2(GM)^2 \kappa_5^2 \Sigma^+_{AB} \sum_{n=0}^{\infty} Z_n(y_+) Z_n(y). \tag{70}$$

We now note that for $e^{-kd} \ll 1$,

$$Z_0(y_+) = \sqrt{k}(1 - e^{-2kd})^{-1/2} \gg Z_n(y_+), \quad n > 0. \tag{71}$$

That is, the $n > 0$ terms in the sum are much smaller than the 0^{th} order contribution. This motivates an approximation where the $n > 0$ terms on the righthand side of (70) are neglected, which is the so-called 'zero-mode truncation'.

When this approximation is enforced, we find that h_{AB} must be proportional to $Z_0(y)$; i.e., there is no contribution to h_{AB} from any of the KK modes. Hence, we have $\hat{\mu}^2 h_{AB} = 0$. The resulting expression has trivial y dependence, so we can freely set $y = y_+$ to obtain the equation of motion for h_{AB} at the *unperturbed* position of the visible brane:

$$\hat{\Delta}_{AB}{}^{CD} h^+_{CD} = -2(GM)^2 \kappa_5^2 \Sigma^+_{AB} Z_0^2(y_+) \tag{72}$$

But we are not really interested in h^+_{AB}, the physically relevant quantity is the perturbation of the induced metric on the perturbed brane, which is defined as the variation of

$$q^+_{AB} = [g_{AB} - n_A n_B]^+. \tag{73}$$

We calculate δq^+_{AB} in the same way as we calculated $\delta Q^\pm_{AB}$ above (except for the fact that q_{AB} shows no explicit dependence on T^+_{AB}):

$$\delta q^+_{AB} = \left\{ \frac{\delta q_{AB}}{\delta \Phi} \delta \Phi + \frac{\delta q_{AB}}{\delta n_C} \delta n_C + \frac{\delta q_{AB}}{\delta g_{CD}} \delta g_{CD} \right\}^+_0. \tag{74}$$

These variations are straightforward, and we obtain:

$$\delta q^+_{AB} \equiv \bar{h}^+_{AB} = h^+_{AB} + 2k\xi^+ q^+_{AB} - (n_A\nabla_B + n_B\nabla_A)\xi^+, \tag{75}$$

where all quantities on the right are evaluated in the background and at the unperturbed position of the brane. Note that $\bar{h}_{AB}n^A \neq 0$, which reflects the fact that n_A is no longer the normal to the brane after perturbation.

We now define the 4-tensors

$$\bar{h}^+_{\alpha\beta} = e^A_\alpha e^B_\beta \bar{h}^+_{AB}, \quad T^+_{\alpha\beta} = e^A_\alpha e^B_\beta T^+_{AB}. \tag{76}$$

Here, $\bar{h}^+_{\alpha\beta}$ is the actual metric perturbation on the visible brane. Note that this perturbation is neither transverse or tracefree:

$$\nabla^\gamma \bar{h}^+_{\gamma\alpha} = 2k\nabla_\alpha\xi^+, \quad g^{\alpha\beta}\bar{h}^+_{\alpha\beta} = 8k\xi^+. \tag{77}$$

We can now re-express the equation of motion (72) in terms of $\bar{h}^+_{\alpha\beta}$ instead of h^+_{AB} using (75). Dropping the $+$ superscripts, we obtain

$$\begin{aligned}\nabla^\gamma\nabla_\gamma\bar{h}_{\alpha\beta} + \nabla_\alpha\nabla_\beta\bar{h}^\gamma{}_\gamma - \nabla^\gamma\nabla_\alpha\bar{h}_{\beta\gamma} - \nabla^\gamma\nabla_\beta\bar{h}_{\alpha\gamma} = \\ -2Z_+^2\kappa_5^2\left[T_{\alpha\beta} - \frac{1}{3}\left(1+\frac{k}{2Z_+^2}\right)T^\gamma{}_\gamma g_{\alpha\beta}\right] + (6k-4Z_+^2)\nabla_\alpha\nabla_\beta\xi,\end{aligned} \tag{78}$$

where we have defined

$$Z_+^2 = Z_0^2(y_+) = k(1-e^{-2kd})^{-1}. \tag{79}$$

In obtaining this expression, we have made use of the ξ equation of motion:

$$g^{\alpha\beta}\nabla_\alpha\nabla_\beta\xi = \tfrac{1}{6}\kappa_5^2 g^{\alpha\beta}T_{\alpha\beta}. \tag{80}$$

Note that we still have the freedom to make a gauge transformation on the brane that involves an arbitrary 4-dimensional coordinate transformation generated by η_α:

$$\bar{h}_{\alpha\beta} \to \bar{h}_{\alpha\beta} + \nabla_\alpha\eta_\beta + \nabla_\beta\eta_\alpha. \tag{81}$$

We can use this gauge freedom to impose the condition

$$\nabla_\beta\bar{h}^\beta{}_\alpha - \tfrac{1}{2}\nabla_\alpha\bar{h}^\beta{}_\beta = (2Z_+^2 - 3k)\nabla_\alpha\xi. \tag{82}$$

Then, the equation of motion for 4-metric fluctuations reads

$$\nabla^\gamma\nabla_\gamma\bar{h}_{\alpha\beta} + 2R_\alpha{}^\gamma{}_\beta{}^\delta\bar{h}_{\gamma\delta} = -16\pi G\left[T_{\alpha\beta} - \left(\frac{1+\omega_{\rm BD}}{3+2\omega_{\rm BD}}\right)T^\gamma{}_\gamma g_{\alpha\beta}\right], \tag{83}$$

where we have identified

$$\omega_{\rm BD} = \frac{3}{2}(e^{2d/\ell}-1), \quad G = \frac{\kappa_5^2}{8\pi\ell(1-e^{-2d/\ell})}. \tag{84}$$

We see that (83) matches the equation governing gravitational waves in a Brans-Dicke theory with parameter ω_{BD}. Hence in the zero-mode truncation, the perturbations of the black string braneworld are indistinguishable from a 4-dimensional scalar tensor theory.

Note that (83) must hold everywhere in our model, so we can consider the situation where our solar system is the perturbative brane matter located somewhere in the extreme far-field region of the black string. The forces between the various celestial bodies will be governed by (83) in the $R_{\alpha\beta\gamma\delta} \approx 0$ limit. In this scenario, solar system tests of general relativity place bounds on the Brans-Dicke parameter, and hence d/ℓ:

$$\omega_{\text{BD}} \gtrsim 4 \times 10^4 \quad \Rightarrow \quad d/\ell \gtrsim 5. \tag{85}$$

This lower bound on the dimensionless brane separation will be an important factor in the discussion below.

6 Beyond the zero-mode truncation

In this section, we specialize to the situation where there is perturbative matter located on one of the branes and no other sources. Unlike §5, our interest here is to predict deviations from general relativity, so we will not use the zero-mode truncation. Just as in 4-dimensional black hole perturbation theory, we introduce the tensor spherical harmonics to further decompose the equations of motion for a given KK mode into polar and axial parts.

6.1 KK mode decomposition

To begin, we make the assumptions

$$\Sigma_{AB}^{\text{bulk}} = 0, \text{ and } \Sigma_{AB}^{+} = 0 \text{ or } \Sigma_{AB}^{-} = 0; \tag{86}$$

i.e., we set the matter perturbation in the bulk and one of the branes equal to zero. Note that due to the linearity of the problem we can always add up solutions corresponding to different types of sources; hence, if we had a physical situation with many different types of matter, it would be acceptable to solve for the radiation pattern induced by each source separately and then sum the results.

We decompose h_{AB} as

$$h_{AB} = \frac{\kappa_5^2 (GM)^2}{\mathscr{C}} e_A^\alpha e_B^\beta \sum_{n=0}^{\infty} Z_n(y) Z_n(y_\pm) h_{\alpha\beta}^{(n)}. \tag{87}$$

Here, $\mathscr{C}$ is a normalization constant (to be specified later) with dimensions of $(\text{mass})^{-4}$, and the expansion coefficients $h_{\alpha\beta}^{(n)}$ are dimensionless. We define a di-

mensionless brane stress-energy tensors and brane bending scalars by

$$\Theta^{\pm}_{\alpha\beta} = \mathscr{C} e^A_\alpha e^B_\beta T^{\pm}_{AB}, \quad \tilde{\xi}^{\pm} = \frac{\mathscr{C}\xi^{\pm}}{(GM)^2\kappa_5^2}. \tag{88}$$

Omitting the $\pm$ superscripts, we find that the equation of motion for $h^{(n)}_{\alpha\beta}$ is

$$(GM)^2\left[\nabla^\gamma\nabla_\gamma h^{(n)}_{\alpha\beta} + 2R_\alpha{}^\gamma{}_\beta{}^\delta h^{(n)}_{\gamma\delta}\right] - \mu_n^2 h^{(n)}_{\alpha\beta} = \\ -2\left(\Theta_{\alpha\beta} - \tfrac{1}{3}\Theta g_{\alpha\beta}\right) - 4(GM)^2\nabla_\alpha\nabla_\beta\tilde{\xi}, \tag{89}$$

while the equation of motion for $\tilde{\xi}$ is

$$\nabla^\alpha\nabla_\alpha\tilde{\xi} = \tfrac{1}{6}\Theta. \tag{90}$$

We also have the conditions

$$\nabla^\alpha h^{(n)}_{\alpha\beta} = \nabla^\alpha\Theta_{\alpha\beta} = 0 = g^{\alpha\beta}h^{(n)}_{\alpha\beta}. \tag{91}$$

Note that in all of these equations, all 4-dimensional quantities are to be calculated with the Schwarzschild metric $g_{\alpha\beta}$. In particular, $\Theta = g^{\alpha\beta}\Theta_{\alpha\beta}$.

6.2 The multipole decomposition

In addition to the decomposition of h_{AB} in terms of KK mode functions, the symmetry of the background geometry dictates that we decompose the problem in terms of spherical harmonics:

$$\tilde{\xi} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_{lm}\tilde{\xi}_{lm}, \tag{92a}$$

$$h^{(n)}_{\alpha\beta} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{i=1}^{10}[Y^{(i)}_{lm}]_{\alpha\beta}\, h_i^{(nlm)}, \tag{92b}$$

$$\Theta_{\alpha\beta} = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{i=1}^{10}[Y^{(i)}_{lm}]_{\alpha\beta}\,\Theta_i^{(lm)}. \tag{92c}$$

Here, $[Y^{(i)}_{lm}]_{\alpha\beta}$ are the tensorial spherical harmonics in 4 dimensions, which are the same quantities that appear in conventional black hole perturbation theory. The tensor harmonics depend only on the angular coordinates $\Omega = (\theta,\phi)$, while the expansion coefficients depend on t and r:

$$\tilde{\xi}_{lm} = \tilde{\xi}_{lm}(t,r), \quad h_i^{(nlm)} = h_i^{(nlm)}(t,r), \quad \Theta_i^{(lm)} = \Theta_i^{(lm)}(t,r). \tag{93}$$

To define the tensor harmonics, first define the orthonormal 4-vectors

$$t^\alpha = f^{-1/2}\partial_t, \quad r^\alpha = f^{1/2}\partial_r, \quad \theta^\alpha = r^{-1}\partial_\theta, \quad \phi^\alpha = (r\sin\theta)^{-1}\partial_\phi. \tag{94}$$

The we define

$$\gamma_{\alpha\beta} = g_{\alpha\beta} + t_\alpha t_\beta - r_\alpha r_\beta = \theta_\alpha\theta_\beta + \phi_\alpha\phi_\beta, \quad t^\alpha\gamma_{\alpha\beta} = r^\alpha\gamma_{\alpha\beta} = 0, \tag{95}$$

which is the projection tensor onto the 2-spheres of constant r and t, and the anti-symmetric tensor $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$

$$\varepsilon_{\alpha\beta} = \theta_\alpha\phi_\beta - \phi_\alpha\theta_\beta. \tag{96}$$

Using these objects, the $[Y^{(i)}_{lm}]_{\alpha\beta}$ are defined in Table 1.[3]

Table 1 The spherical tensor harmonics $[Y^{(i)}_{lm}]_{\alpha\beta}$

index i	Polar harmonics $\mathbb{P}^{ilm}_{\alpha\beta}$	Axial harmonics $\mathbb{A}^{ilm}_{\alpha\beta}$
1	$f^{-1}t_\alpha t_\beta Y_{lm}$	$2f^{-1/2}t_{(\alpha}\varepsilon_{\beta)\gamma}\nabla^\gamma Y_{lm}$
2	$2t_{(\alpha}r_{\beta)}Y_{lm}$	$2f^{+1/2}r_{(\alpha}\varepsilon_{\beta)\gamma}\nabla^\gamma Y_{lm}$
3	$fr_\alpha r_\beta Y_{lm}$	$\gamma_{\gamma(\alpha}\varepsilon_{\beta)\delta}\nabla^\delta\nabla^\gamma Y_{lm}$
4	$-2t_{(\alpha}\gamma_{\beta)\gamma}\nabla^\gamma Y_{lm}$	$\cdots$
5	$+2r_{(\alpha}\gamma_{\beta)\gamma}\nabla^\gamma Y_{lm}$	$\cdots$
6	$r^{-2}\gamma_{\alpha\beta}Y_{lm}$	$\cdots$
7	$\gamma_{\alpha\gamma}\gamma_{\beta\delta}\nabla^\gamma\nabla^\delta Y_{lm}$	$\cdots$

Notice that we have divided the ten tensor harmonics into two groups labeled 'polar' and 'axial'. This division is based on how they transform under the parity, or space-inversion, operation $\mathbf{r} \to -\mathbf{r}$. In particular, under this type of operation, polar objects acquire a $(-1)^l$ factor, while axial quantities transform as $(-1)^{l+1}$.[4] It is useful to re-write the spherical harmonic decomposition of $h^{(n)}_{\alpha\beta}$ in terms of explicitly polar and axial parts:[5]

$$h^{(n)}_{\alpha\beta} = \underbrace{\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{i=1}^{7}\mathbb{P}^{ilm}_{\alpha\beta}(\boldsymbol{\Omega})\,\mathscr{P}^{(n)}_{ilm}(t,r)}_{\text{polar contribution } h^{(n,\text{polar})}_{\alpha\beta}} + \underbrace{\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{i=1}^{3}\mathbb{A}^{ilm}_{\alpha\beta}(\boldsymbol{\Omega})\,\mathscr{A}^{(n)}_{ilm}(t,r)}_{\text{axial contribution } h^{(n,\text{axial})}_{\alpha\beta}}. \tag{97}$$

[3] The definition of tensor harmonics is not unique; there are numerous other conventions in the literature.

[4] Alternatively, we can note that any tensor harmonic whose definition involves the pseudo-tensor ε_{ab} is automatically an axial object.

[5] A similar decomposition for $\Theta_{\alpha\beta}$ also exists.

In this expression and similar ones below, there is no summation over the spherical harmonic or i index unless indicated explicitly.

It is easy to confirm that the parity operation commutes with the $\hat{\Delta}_{AB}{}^{CD}$ and $\hat{\mu}^2$ operators in (44), or conversely commutes with the operator $\delta_\alpha^\gamma \delta_\beta^\delta \nabla^\lambda \nabla_\lambda + 2R_\alpha{}^\gamma{}_\beta{}^\delta$ in (89). Therefore, solutions of (89) that are eigenfunctions of the parity operator with different eigenvalues are decoupled from one another; i.e., we can solve for the dynamics of $h_{\alpha\beta}^{(n,\text{polar})}$ and $h_{\alpha\beta}^{(n,\text{axial})}$ individually. As is common for spherically symmetric systems, modes with different values of l and m are also decoupled.

Before moving on, we should mention that the decomposition of the brane bending scalar $\tilde{\xi}$ is given entirely in terms of Y_{lm}; i.e., it is an explicitly polar quantity. It follows that $\nabla_\alpha \nabla_\beta \tilde{\xi}$ is also a polar quantity, which means that the brane bending contribution in (89) only sources polar GW radiation.

7 Homogeneous axial perturbations

In this section, we present the equations of motion for the axial moments of $h_{\alpha\beta}^{(n)}$ in the absence of all matter sources. As mentioned above, the brane bending contribution to (89) is a polar quantity. Therefore, the axial GW modes are completely decoupled from the brane bending scalar. Hence, the equation we try to solve in this section is simply:

$$(GM)^2 \left[\nabla^\gamma \nabla_\gamma h_{\alpha\beta}^{(nlm,\text{axial})} + 2R_\alpha{}^\gamma{}_\beta{}^\delta h_{\gamma\delta}^{(nlm,\text{axial})}\right] - \mu_n^2 h_{\alpha\beta}^{(nlm,\text{axial})} = 0, \tag{98}$$

where the total axial contribution to $h_{\alpha\beta}^{(n)}$ is

$$h_{\alpha\beta}^{(n,\text{axial})} = \sum_{lm} h_{\alpha\beta}^{(nlm,\text{axial})}. \tag{99}$$

In addition to this equation, remember that we also need to satisfy the gauge conditions (91).

Notice that (98) reduces to the graviton equation of motion in ordinary GR for $m_n = 0$, which corresponds to $n = 0$. It turns out that the $n = 0$ case must be handled separately from the $n \geq 0$ case due to an enhanced gauge symmetry present in the zero-mode sector. Therefore, for the purposes of this section we always assume $n \geq 0$.

7.1 High angular momentum $l \geq 2$ radiation

In Table 1, notice that the axial harmonics are identically equation to zero for $l = 0$. Also note that for $l = 1$, the third harmonic vanishes $\mathbb{A}_{lm}^{3lm} = 0$. This means that there

are no axial harmonics for $l = 0$ and that $l = 1$ is a special case. In this subsection, we concentrate on the $l \geq 2$ situation, where all of the axial tensor harmonics are non-trivial.

The decomposition of $h^{(nlm,\text{axial})}_{\alpha\beta}$ explicitly reads

$$h^{(nlm,\text{axial})}_{\alpha\beta} = \mathbb{A}^{1lm}_{\alpha\beta}(\Omega)\,\mathscr{A}^{(n)}_{1lm}(t,r) + \mathbb{A}^{2lm}_{\alpha\beta}(\Omega)\,\mathscr{A}^{(n)}_{2lm}(t,r) + \mathbb{A}^{3lm}_{\alpha\beta}(\Omega)\,\mathscr{A}^{(n)}_{3lm}(t,r). \quad (100)$$

When this is substituted into the equation of motion (98) and gauge conditions (91), we get four PDEs that must be satisfied by the three expansion coefficients. These four equations are not independent, however, as the time derivative of one of them is a linear combination of the other three. Removing this equation, it is possible to use one of the other PDEs to algebraically eliminate $\mathscr{A}^{(n)}_{1lm}$ from the other two equations. Defining the 'master variables'

$$u_{nlm}(t,r) = f(r)\mathscr{A}^{(n)}_{2lm}(t,r), \quad v_{nlm}(t,r) = r^{-1}\mathscr{A}^{(n)}_{3lm}(t,r), \quad (101)$$

we eventually find that

$$0 = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2}\right)\begin{pmatrix} u_{nlm} \\ v_{nlm} \end{pmatrix} + \mathbf{V}_{nl}\begin{pmatrix} u_{nlm} \\ v_{nlm} \end{pmatrix}. \quad (102)$$

Here, $\mathbf{V}_{nl}$ is a potential matrix, given by

$$\mathbf{V}_{nl} = f\begin{pmatrix} \frac{5f}{r^2} + \frac{f''}{2} - \frac{2f'}{r} + \frac{l(l+1)-1}{r^2} + m_n^2 & \frac{f'[2-l(l+1)]}{2r} \\ \frac{4}{r^2} & \frac{f'}{r} + \frac{l(l+1)-2}{r^2} + m_n^2 \end{pmatrix}, \quad (103)$$

and the well-known tortoise coordinate is defined by

$$r_* = r + 2GM\ln\left(\frac{r}{2GM} - 1\right). \quad (104)$$

Hence, to describe homogeneous axial perturbations of the black string braneworld, one needs to specify initial data for u_{nlm} and v_{nlm}, solve the coupled wave equations (102), and then use the definitions (109) to obtain the original expansion coefficients $\mathscr{A}^{(n)}_{2lm}$ and $\mathscr{A}^{(n)}_{3lm}$. The last step it to integrate one of the original equations of motion,

$$\frac{\partial\mathscr{A}^{(n)}_{1lm}}{\partial t} = f^2\frac{\partial\mathscr{A}^{(n)}_{2lm}}{\partial r} + \frac{f(2f+f'r)}{r}\mathscr{A}^{(n)}_{2lm} + \frac{f[l(l+1)-2]}{2r^2}\mathscr{A}^{(n)}_{3lm}, \quad (105)$$

to obtain the other expansion coefficient $\mathscr{A}^{(n)}_{1lm}$. This procedure can be repeated for each individual value of n, l, and m. However, it should be noted that since the potential matrix does not explicitly depend on m, solutions that share the same values of n and l only really differ from one another by the choice of initial data.

Why are we interested in solving homogeneous problems like the one presented in this section? Recall that in the case of 4-dimensional black hole perturbation theory, the numeric solution of the homogeneous axial wave equation lead to the

discovery of quasinormal modes. In other words, by examining the solutions of equations such as (102), one can learn a lot about the characteristic behaviour of a system when perturbed away from equilibrium, which is what we shall do in §7.3. The solution of the homogeneous problem can also have some direct observational significance, since it can describe how the system settles down into its equilibrium state after some event. That is, we expect the late time axial gravitational wave signal from a black string to be described by the solutions of (102) after a black string is formed or undergoes some traumatic event.

Before moving on, it is worthwhile to note the asymptotic behaviour of the potential matrix:

$$\lim_{r_*\to-\infty}\mathbf{V}_{nl}=0,\quad \lim_{r\to+\infty}\mathbf{V}_{nl}=\begin{pmatrix} m_n^2+\mathscr{O}(r^{-1}) & \mathscr{O}(r^{-3})\\ \mathscr{O}(r^{-2}) & m_n^2+\mathscr{O}(r^{-1})\end{pmatrix}. \tag{106}$$

For $r_*\to-\infty$, which corresponds to the black hole horizon, we see that u_{nlm} and v_{nlm} behave as free massless scalars. Conversely, far away from the black hole they behave as decoupled scalars of mass m_n. It turns out that the asymptotic form of $\mathbf{V}_{nl}$ as $r\to\infty$ is crucial in determining the characteristic GW signal from a black string, as we will see below.

7.2 Axial *p*-waves

For the sake of completeness, we can write down the equations of motion governing the $l=1$, or p-wave, sector. In this case, general fluctuations are described by

$$h_{\alpha\beta}^{(n1m,\text{axial})}=\mathbb{A}_{\alpha\beta}^{1,1,m}(\Omega)\,\mathscr{A}_{1,1,m}^{(n)}(t,r)+\mathbb{A}_{\alpha\beta}^{2,1,m}(\Omega)\,\mathscr{A}_{2,1,m}^{(n)}(t,r). \tag{107}$$

In this case, when we substitute this into the equation of motion (98), we find a single master equation

$$0=(\partial_t^2-\partial_{r_*}^2)u_{n1m}+V_{n1}u_{n1m}, \tag{108}$$

where

$$u_{n1m}(t,r)=f(r)\mathscr{A}_{2,1,m}^{(n)}(t,r), \tag{109}$$

and the potential is

$$V_{n1}=f\left(\frac{5f+1}{r^2}-\frac{2f'}{r}+\frac{f''}{2}+m_n^2\right). \tag{110}$$

Once this equation is solved and $\mathscr{A}_{2,1,m}^{(n)}$ is found, the remaining expansion coefficient is determined by a quadrature:

$$\frac{\partial\mathscr{A}_{1,1,m}^{(n)}}{\partial t}=f^2\frac{\partial\mathscr{A}_{2,1,m}^{(n)}}{\partial r}+\frac{f(2f+f'r)}{r}\mathscr{A}_{2,1,m}^{(n)}. \tag{111}$$

Notice that this is identical to (105) with $l = 1$.

One comment on the $l = 1$ perturbations is in order before we proceed. In ordinary black hole perturbation theory, there are no truly time-dependent p-wave perturbations of the Schwarzschild spacetime. This is because the $l = 1$ perturbations correspond to giving the black hole a small amount of angular momentum about some axis in 3-space; i.e., they represent the linearization of the Kerr solution about the Schwarzschild background, and are hence time-independent. In the black string case, however, the $l = 1$ perturbation can be viewed as endowing a small spin to the Schwarzschild 4-metrics on each Σ_y hypersurface. However, the amount of spin delivered to each hypersurface by each massive mode is not uniform, in fact it is easily shown that it is proportional to $Z_n(y)$ evaluated at that hypersurface. In other words, dipole perturbations give rise to a differentially rotating black string, where the amount of rotation varies with y. It turns out that there is no time-independent black string solution of this type, so we have dynamic perturbations. The exception is the zero mode $n = 0$, which gives rise to a uniform rotation of the black string; i.e., these perturbations give rise to the linearization of (18) about (15).

7.3 Numeric integration of quadrupole equations

In Figure 2, we present the results of some numerical solutions of equation (102) for the case of quadrupole radiation $l = 2$. In this plot, we assume that we have Gaussian initial data for u_{n2m} on some initial time slice and that $v_{n2m} = 0$ initially. It turns out that the particular choice of initial data does not much affect the outcome of the simulations; that is, changing the shape or location of the initial Gaussian, or taking $v_{n2m} \neq 0$, results in very similar waveforms.

The key feature of the displayed waveforms is the nature of the late time signal. We see that each of the $n > 0$ waveforms exhibits very long-lived late time oscillations.[6] This behaviour is totally unlike the standard picture of black hole oscillations in GR, where one expects the late time ringdown waveform to be a featureless power law tail. This kind of signal is exhibited by the $n = 0$ zero-mode signal, which we already know corresponds exactly to the GR result. One of the most remarkable things about the massive mode signal is that it is present for all types of initial data, suggesting that it is a fundamental property of the black string as opposed to just some simulation fluke. In this sense the massive mode tail observed here is analogous to the quasinormal modes of standard 4-dimensional theory.

An exercise in curve-fitting reveals that the late time massive signal is well modeled by

$$\left\{ \begin{matrix} u_{n2m} \\ v_{n2m} \end{matrix} \right\} \sim \text{const} \times \left(\frac{t}{GM} \right)^{-5/6} \sin(m_n t + \phi). \tag{112}$$

That is, the frequency of oscillation matches the mass of the mode. The decay rate $\sim t^{-5/6}$ is much slower than the decay of the zero-mode signal, which decays at

[6] A mathematical rationalization of this is given is §10.2.2.

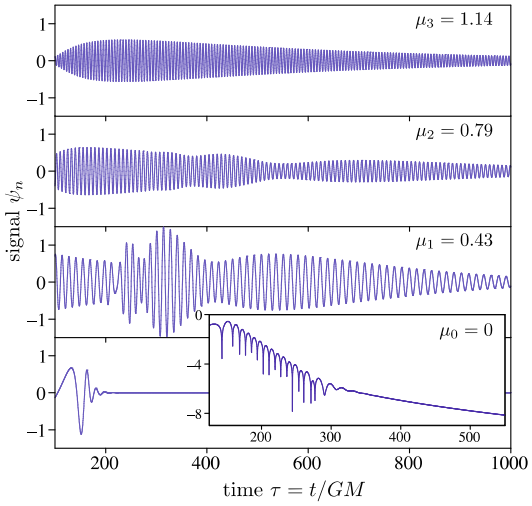

Fig. 2 Results of the integration of the quadropole axial equations of motion. The waveforms are observed at $r_* = 100GM$ while the initial data was originally located at $r_* = 50GM$. We show results for the $n = 0, 1, 2, 3$ modes. The massive mode signals are characterized by a long-lasting oscillating tail; i.e. u_{n2m} and v_{n2m} are proportional to $(t/GM)^{-5/6}\sin(m_n t + \phi)$ at late times for $n > 0$ (here, ϕ is a phase angle). This is in contrast to the zero-mode result, which shows no oscillations and a power law decay at late times. (The inset shows the zero mode result on a log-log scale.)

least as fast as t^{-4}. We can confirm via simulations that these result holds for other values of l. Hence, we are lead to the following important conclusion:

> Irrespective of the initial amplitudes of the various KK modes, if one waits long enough the GW signal from a perturbed black string will be dominated by a superposition of slowly-decaying massive modes.

A challenge for gravitational wave astronomy is to observe these massive mode signals directly. The actual prospects of doing this are discussed in §10.4.

8 Spherical perturbations with source terms

We can re-write the decomposition (113) by explicitly pulling out the spherical contributions:

$$\tilde{\xi} = \frac{\xi^{(s)}}{\sqrt{4\pi}} + \sum_{l=1}^{\infty}\sum_{m=-l}^{l} Y_{lm}\tilde{\xi}_{lm}, \tag{113a}$$

$$h^{(n)}_{\alpha\beta} = \frac{h^{(n,s)}_{\alpha\beta}}{\sqrt{4\pi}} + \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\sum_{i=1}^{10} [Y^{(i)}_{lm}]_{\alpha\beta}\, h^{(nlm)}_{i}, \tag{113b}$$

$$\Theta_{\alpha\beta} = \frac{\Theta^{(s)}_{\alpha\beta}}{\sqrt{4\pi}} + \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\sum_{i=1}^{10} [Y^{(i)}_{lm}]_{\alpha\beta}\, \Theta^{(lm)}_{i}. \tag{113c}$$

Here, $\xi^{(s)}$, $h^{(n,s)}_{\alpha\beta}$ and $\Theta^{(s)}_{\alpha\beta}$ represent the spherically-symmetric parts of the brane-bending scalar, metric perturbation, and brane stress-energy tensor, respectively. In this section, we are going to concentrate on the dynamics of this sector when there are non-trivial matter sources on one of the branes sourcing gravitational radiation. The reason that we focus on the $l = 0$, or *s*-wave, sector is computational convenience; the equations of motion become rather involved for higher multipoles.

Before starting to calculate things, we note that some readers may be a little confused as to why we are even looking at spherically-symmetric gravitational radiation. In general relativity, it is a well-known consequence of Birkhoff's theorem that there is no spherically-symmetric radiation about a Schwarzschild black hole. This is because the theorem states that the only solutions to the Einstein equations with cosmological constant with structure $\mathbb{R}^2 \times S_d$ are $(d+2)$-dimensional Schwarzschild-de Sitter or Schwarzschild anti-de Sitter black holes. Since these are static solutions, any perturbation that respects the S_d symmetry of the background must also be static.[7] But the black string background has structure $(\mathbb{R}^2 \times S_2) \times S/\mathbb{Z}_2$. Birkhoff's theorem does not apply in this case and we can indeed have time-dependant solutions of $G_{AB} = 6k^2 g_{AB}$ with the same structure. Therefore, it is possible to have dynamical spherically-symmetric radiation around a black string, which is what we study in this section.

8.1 Spherical master variables

We write the $l = 0$ contribution to the metric perturbation as

$$h^{(n,s)}_{\alpha\beta} = \mathsf{H}_1\, t_\alpha t_\beta - 2\mathsf{H}_2\, t_{(\alpha} r_{\beta)} + \mathsf{H}_3\, r_\alpha r_\alpha + \mathsf{K}\,\gamma_{\alpha\beta}, \tag{114}$$

where the 4-vectors and $\gamma_{\alpha\beta}$ are defined in (94) and (95), respectively. Each of the expansion coefficients is a function of t and r; i.e., $\mathsf{H}_i = \mathsf{H}_i(t,r)$ and $\mathsf{K} = \mathsf{K}(t,r)$. Notice that the condition that $h^{(n,s)}_{\alpha\beta}$ is tracefree implies

$$\mathsf{K} = \tfrac{1}{2}(\mathsf{H}_1 - \mathsf{H}_3). \tag{115}$$

[7] Static here means that one can find a gauge in which the perturbation does not depend on time.

Before going further, it is useful to define dimensionless coordinates:

$$\rho = \frac{r}{GM}, \quad \tau = \frac{t}{GM}, \quad x = \rho + 2\ln\left(\frac{\rho}{2} - 1\right). \tag{116}$$

Then, when our decompositions (113) are substituted into the equations of motion, we find that all components of the metric perturbation are governed by master variables

$$\psi = \frac{2\rho^3}{2+\mu_n^2\rho^3}\left(\rho\frac{\partial \mathsf{K}}{\partial\tau} - f\mathsf{H}_2\right), \quad \varphi = \rho\frac{\partial\xi^{(s)}}{\partial\tau}. \tag{117}$$

Both $\psi = \psi(\tau, x)$ and $\varphi = \varphi(\tau, x)$ satisfy simple wave equations:

$$(\partial_\tau^2 - \partial_x^2 + V_\psi)\psi = \mathscr{S}_\psi + \hat{\mathscr{I}}\,\varphi, \tag{118a}$$

$$(\partial_\tau^2 - \partial_x^2 + V_\varphi)\varphi = \mathscr{S}_\varphi. \tag{118b}$$

The potential and matter source term in the ψ equation are:

$$V_\psi = \frac{f}{\rho^3\,(2+\rho^3{\mu_n}^2)^2}\Big[{\mu_n}^6\rho^9 + 6\,{\mu_n}^4\rho^7 - 18\,{\mu_n}^4\rho^6 - 24\,{\mu_n}^2\rho^4 + 36\,{\mu_n}^2\rho^3 + 8\Big], \tag{119a}$$

$$\mathscr{S}_\psi = \frac{2f\rho^3}{3(2+\mu_n^2\rho^3)^2}\Big[\rho(2+\mu_n^2\rho^3)\partial_\tau(2\Lambda_1 + 3\Lambda_3) + 6(\mu_n^2\rho^3 - 4)\,f\Lambda_2\Big]. \tag{119b}$$

Here, we have defined the following three scalars derived from the dimensionless stress-energy tensor $\Theta^{(s)}_{\alpha\beta}$:

$$\Lambda_1 = -\Theta^{(s)}_{\alpha\beta}g^{\alpha\beta}, \quad \Lambda_2 = -\Theta^{(s)}_{\alpha\beta}t^\alpha r^\beta, \quad \Lambda_3 = +\Theta^{(s)}_{\alpha\beta}\gamma^{\alpha\beta}. \tag{120}$$

The potential and source terms in the brane-bending equation are somewhat less involved:

$$V_\varphi = \frac{2f}{\rho^3}, \quad \mathscr{S}_\varphi = \frac{\rho f}{6}\partial_\tau\Lambda_1. \tag{121}$$

Finally, the interaction operator is

$$\hat{\mathscr{I}} = \frac{8f}{(2+\mu_n^2\rho^3)^2}\left[6f\rho^2\partial_\rho + (\mu_n^2\rho^3 - 6\rho + 8)\right]. \tag{122}$$

8.2 Inversion formulae

Assuming that we can solve the wave equations (118) for a given source, we need formulae that allow us to express H_i, K in terms of ψ and φ in order to make gravitational wave prediction. This can be derived by inverting the master variable definitions (117) with the aid (118). The general formulae are actually very complicated and not particularly enlightening, so we do not reproduce them here. Ultimately, to make observational predictions it is sufficient to know the form of the metric perturbation far away from the black string and the matter sources, so we evaluate the general inversion formulae in the limit of $\rho \to \infty$ and with $\Lambda_i = 0$:

$$\begin{aligned}
\partial_\tau \mathsf{H}_1 &= \frac{1}{\rho}\left[\left(\partial_\tau^2 + \frac{3}{\rho}\partial_\rho + \mu_n^2\right)\psi + \frac{4}{\mu_n^2}\left(\partial_\tau^2 - \frac{1}{\rho}\partial_\rho\right)\varphi\right], \\
\mathsf{H}_2 &= \frac{1}{\rho}\left[\left(\partial_\rho + \frac{2}{\rho}\right)\psi + \frac{4}{\mu_n^2}\left(\partial_\rho - \frac{1}{\rho}\right)\varphi\right], \\
\partial_\tau \mathsf{H}_3 &= \frac{1}{\rho}\left[\left(\partial_\tau^2 + \frac{1}{\rho}\partial_\rho\right)\psi + \frac{4}{\mu_n^2}\left(\partial_\tau^2 - \frac{2}{\rho}\partial_\rho\right)\varphi\right], \\
\partial_\tau \mathsf{K} &= \frac{1}{\rho}\left[\left(\frac{1}{\rho}\partial_\rho + \frac{\mu_n^2}{2}\right)\psi + \frac{4}{\mu_n^2\rho}\left(\partial_\rho - \frac{1}{\rho}\right)\varphi\right].
\end{aligned} \tag{123}$$

Note that these do not actually complete the inversion; in most cases, a quadrature is also required to arrive at the final form of the metric perturbation.

8.3 The Gregory-Laflamme instability

We now discuss one extremely important consequence of the equation of motion (118). Note that we can always add-on a solution of the homogeneous wave equation:

$$0 = (\partial_\tau^2 - \partial_x^2 + V_\psi)\psi, \tag{124}$$

to any particular solution ψ_p of (118a) generated by a given source. If we analyze this homogenous equation in Fourier space by setting $\psi(\tau, x) = e^{i\omega\tau}\Psi(x)$, we find that

$$\omega^2 \Psi = -\frac{d^2\Psi}{dx^2} + V_\psi \Psi. \tag{125}$$

This is identical to the time-independent Schrödinger equation from elementary quantum mechanics with ω^2 playing the role of the energy parameter. Now, suppose that the potential supports a bound state solution with negative energy $\omega^2 < 0$. That is, suppose we can find a solution of this ODE with $\Psi \to 0$ as $x \to \pm\infty$ with $\omega = -i\Gamma$, where $\Gamma > 0$. In such cases, $\psi \propto e^{\Gamma t}$ and we have an exponentially growing solution to the equations of motion, which represents a linear instability of the

system. Since such a tachyonic mode ψ is spatially bounded and arbitrary small in the past, it is possible for any initial data with compact support to excite it.

Clearly, the black string braneworld cannot be a viable black hole model if we can find such a tachyonic mode. It turns out that the potential V_ψ (119a) is not actually capable of supporting a negative energy bound state for all values of μ. There are numerous ways of demonstrating this; including the WKB method and direct numeric solution of (125). One finds that no bound state exists if

$$\mu_n > \mu_c \approx 0.4301 \text{ or } \mu_n = 0. \tag{126}$$

That is, the zero-mode of the *s*-wave sector is stable[8], and the high-mass modes are also stable. This implies that the black string braneworld is perturbatively stable if the smallest KK mass satisfies

$$\mu_1 = GMm_1 > \mu_c \approx 0.4301. \tag{127}$$

Under the approximation that the first mode is light ($x_1 e^{-kd} \ll 1$) and using $G = \ell_{\rm Pl}/M_{\rm Pl}$, this gives a restriction on the black string mass

$$\frac{M}{M_{\rm Pl}} \gtrsim \frac{\ell}{\ell_{\rm Pl}} \frac{\mu_c}{x_1} e^{kd}, \tag{128}$$

or equivalently,

$$\frac{M}{M_\odot} \gtrsim 8 \times 10^{-9} \left(\frac{\ell}{0.1 \text{ mm}} \right) e^{d/\ell}. \tag{129}$$

If we take $\ell = 0.1$ mm, then we see that all solar mass black holes will in actuality be stable black strings provided that $d/\ell \lesssim 19$. The stability of the black string braneworld is summarized in Figure 3.

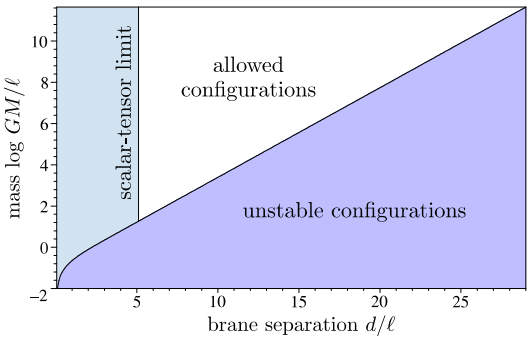

Fig. 3 The stability of the black string braneworld model. If the black string mass M, or the brane separation d is selected such that GM/ℓ and d/ℓ lies outside of the 'unstable configurations' configurations portion of parameter space, the model is stable. We have also indicated the $d/\ell \gtrsim 5$ limit imposed by the low energy scalar-tensor limit of the model in the solar system (c.f. §5).

Before moving on, we have two final comments: First, we should note that all black strings are unstable if the distance between the branes becomes large $d \to \infty$.

[8] One can show that this is actually a gauge mode

This essentially means that there is no stable black string solution when the extra dimension is infinite. This is the well known Gregory-Laflamme instability of black strings (7; 8). Second, if we denote the minimum mass stable black string to be $M_{\rm GL}$ for a given d/ℓ, note that we do note claim that black holes with $M < M_{\rm GL}$ do not exist in this braneworld setup. Rather, such small mass black holes are not described by the black string bulk. They would instead be described by some localized black hole solution that has yet to be obtained. It has been suggested in the literature that the transition between the localized black hole and black string may be a violent first order phase transition, an hence be a significant source of gravitational radiation (12).

9 Point particle sources on the brane

Up until this point, we have either been discussing homogenous equations or generic sources. As an illustration of a more specific application of the formulae we have derived, we specialize to the situation where the perturbing brane matter is a 'point particle' located on one of the branes. Our goal is to explicitly write down the equations of motion for the GWs emitted by the particle. This is a situation of a significant astrophysical interest in 4-dimensions, because it is thought to be a good model of 'extreme-mass-ratio-inspirals' (EMRIs). This is a scenario when an object of mass M_p merges with a black hole of mass M. When $M_p \ll M$, it is a good approximation to replace the small body with a point particle, or delta-function, source. Our interest here is to generalize this standard 4-dimensional calculation to the black string background.

One caution is in order before we proceed: It is not entirely clear that the delta-function approximation is a good one to make in the braneworld scenario. In 4 dimensions, there are only two length scales in the problem: the two Schwarzschild radii $2GM$ and $2GM_p$.[9]. Hence, an extreme scenario is well defined when one scale is much larger than the other. However, in the braneworld scenario there is an additional length scale ℓ. In typical situations, $\ell \ll 2GM_p \ll 2GM$. It is unclear whether or not it is valid to model the perturbing body as a point particle in this case, since a point particle always has a physical size less than ℓ. However, it the absence of a better source model, we will pursue the point particle description here, while always keeping this caveat in mind.

9.1 Point particle stress-energy tensor

We take the particle Lagrangian density to be

[9] We generally consider cases where the physical size of the perturbing particle is close to its horizon radius, as for neutron stars, etc.

$$\mathscr{L}_p^{\pm} = \frac{M_p}{2}\left\{\int \frac{\delta^4(z^\mu - z_p^\mu)}{\sqrt{-q}} q_{\alpha\beta}\frac{dz_p^\alpha}{d\eta}\frac{dz_p^\beta}{d\eta}d\eta\right\}^{\pm}. \tag{130}$$

In this expression, η is a parameter along the particle's trajectory as defined by the $q_{\alpha\beta}$ metric, z_p^μ are the 4 functions describing the particle's position on the brane, and M_p is the particle's mass parameter. Using (13, we find the stress-energy tensor

$$T_{\alpha\beta}^{\pm} = M_p\left\{\int \frac{\delta^4(z^\mu - z_p^\mu)}{\sqrt{-q}} q_{\alpha\rho}q_{\beta\lambda}\frac{dz_p^\rho}{d\eta}\frac{dz_p^\lambda}{d\eta}d\eta\right\}^{\pm}. \tag{131}$$

The contribution from the particle to the total action is

$$S_p^{\pm} = \frac{1}{2}\int_{\Sigma^{\pm}} \mathscr{L}_p^{\pm} = \frac{M_p}{4}\int q_{\alpha\beta}^{\pm}\frac{dz_p^\alpha}{d\eta}\frac{dz_p^\beta}{d\eta}d\eta. \tag{132}$$

Varying this with respect to the trajectory z_p^α and demanding that η is an affine parameter yields that the particle follows a geodesic along the brane:

$$\frac{d^2z_p^\alpha}{d\eta^2} + \Gamma_{\beta\gamma}^{\alpha}[q^{\pm}]\frac{dz_p^\beta}{d\eta}\frac{dz_p^\gamma}{d\eta} = 0, \quad -1 = q_{\alpha\beta}^{\pm}\frac{dz_p^\alpha}{d\eta}\frac{dz_p^\beta}{d\eta}, \tag{133}$$

where $\Gamma_{\beta\gamma}^{\alpha}[q^{\pm}]$ are the Christoffel symbols defined with respect to the $q_{\alpha\beta}^{\pm}$ metric.

We note that the above formulae make explicit use of the induced brane metrics $q_{\alpha\beta}^{\pm}$. However, all of our perturbative formalism is in terms of the Schwarzschild metric $g_{\alpha\beta}$, especially the definition of the Λ_i scalars (120). Hence, it is useful to translate the above expressions using the following definitions:

$$\eta = a_{\pm}\lambda, \quad u^\alpha = \frac{dz_p^\alpha}{d\lambda}, \quad -1 = g_{\alpha\beta}u^\alpha u^\beta. \tag{134}$$

Then, the stress-energy tensor and particle equation of motion become

$$T_{\alpha\beta}^{\pm} = \frac{M_p}{a_{\pm}}\int \frac{\delta^4(z^\mu - z_p^\mu)}{\sqrt{-g}} u_\alpha u_\beta\, d\lambda, \quad u^\alpha\nabla_\alpha u^\beta = 0. \tag{135}$$

Note that the only difference between the stress-energy tensors on the positive and negative tension branes is an overall division by the warp factor.

By switching over to dimensionless coordinates, transforming the integration variable to τ from λ, and making use of the spherical harmonic completeness relationship, we obtain

$$T_{\alpha\beta}^{\pm} = \frac{f}{\mathscr{C}_{\pm}E\rho^2}u_\alpha u_\beta\,\delta(\rho-\rho_p)\left[\frac{1}{4\pi} + \sum_{l=1}^{\infty}\sum_{m=-l}^{l} Y_{lm}(\Omega)Y_{lm}^*(\Omega_p)\right]. \tag{136}$$

Here, we have defined

$$\mathscr{C}_\pm = \frac{(GM)^3}{M_p e^{ky_\pm}}, \quad E = -g_{\alpha\beta}u^\alpha \xi^\beta_{(t)}, \quad \xi^\alpha_{(t)} = \partial_t. \tag{137}$$

As usual, E is the particle's energy per unit rest mass defined with respect to the timelike Killing vector $\xi^\alpha_{(t)}$.

9.2 The *s*-wave sector

Comparing (88) and (113c) with (136), we see that

$$\Theta^{(s)}_{\alpha\beta} = \frac{f}{\sqrt{4\pi}E\rho^2} u_\alpha u_\beta \, \delta[\rho - \rho_p(\tau)], \tag{138a}$$

$$\Lambda_1 = \frac{f}{\sqrt{4\pi}E\rho^2} \, \delta[\rho - \rho_p(\tau)], \tag{138b}$$

$$\Lambda_2 = \frac{E\dot{\rho}_p}{\sqrt{4\pi}f\rho^2} \, \delta[\rho - \rho_p(\tau)], \tag{138c}$$

$$\Lambda_3 = \frac{f\tilde{L}^2}{\sqrt{4\pi}E\rho^4} \, \delta[\rho - \rho_p(\tau)], \tag{138d}$$

where $\dot{\rho}_p = d\rho_p/d\tau$. Here, we have identified L as the total angular momentum of the particle (per unit rest mass), defined by

$$\frac{L^2}{r^2} = \gamma_{\alpha\beta}u^\alpha u^\beta, \quad \tilde{L} = \frac{L}{GM}. \tag{139}$$

Note that for particles traveling on geodesics, E and L are constants of the motion. These are commonly re-parameterized in terms of the eccentricity e and the semi-latus rectum p, both of which are non-negative dimensionless numbers:

$$\begin{aligned} E^2 &= \frac{(p-2-2e)(p-2+2e)}{p\,(p-3-e^2)}, \\ \tilde{L}^2 &= \frac{p^2}{p-3-e^2}. \end{aligned} \tag{140}$$

The orbit can then be conveniently described by the alternative radial coordinate χ, which is defined by

$$\rho = \frac{p}{1+e\cos\chi}. \tag{141}$$

Taking the plane of motion to be $\theta = \pi/2$, we obtain two first order differential equations governing the trajectory

$$\frac{d\chi}{d\tau} = \left[\frac{(p-2-2e\cos\chi)^2(p-6-2e\cos\chi)}{\rho_p^4(p-2-2e)(p-2+2e)}\right]^{1/2},$$
$$\frac{d\phi}{d\tau} = \left[\frac{p\,(p-2-2e\cos\chi)^2}{\rho_p^4(p-2-2e)(p-2+2e)}\right]^{1/2}. \tag{142}$$

These are well-behaved thorough turning points of the trajectory $d\rho_p/dt = 0$. When $e < 1$ we have bound orbits such that $p/(1+e) < \rho_p < p/(1-e)$, while for $e > 1$ we have unbound 'fly-by' orbits whose closest approach is $\rho_p = p/(1+e)$. To obtain orbits that cross the future event horizon of the black string, one needs to apply a Wick rotation to the eccentricity $e \mapsto ie$ and make the replacement $\chi \mapsto i\chi + \pi/2$. Then a radially infalling particle corresponds to $e = \infty$.

It is worthwhile to write out the associated source terms in the wave equation explicitly as a function of orbital parameters

$$\mathscr{S}_\psi = \frac{2f^2\dot{\rho}_p}{3\sqrt{4\pi}E(2+\mu^2\rho^3)}\left[-(2\rho^2+3\tilde{L}^2)\delta'[\rho-\rho_p(\tau)]\right.$$
$$\left.+\frac{6\rho E^2}{f}\left(\frac{\mu^2\rho^3-4}{\mu^2\rho^3+2}\right)\delta[\rho-\rho_p(\tau)]\right],$$
$$\mathscr{S}_\varphi = -\frac{f^2\dot{\rho}_p}{6\sqrt{4\pi}E\rho}\delta'[\rho-\rho_p(\tau)]. \tag{143}$$

Note that

$$|\dot{\rho}_p| < f,$$
$$\dot{\rho}_p = 0 \Rightarrow \mathscr{S}_\psi = \mathscr{S}_\varphi = 0, \tag{144}$$
$$E \gg 1 \Rightarrow \mathscr{S}_\psi \gg \mathscr{S}_\varphi.$$

That is, the particle's speed is always less than unity, the sources wave equation vanish if the particle is stationary or in a circular orbit, and high-energy trajectories imply that the system's dynamics are not too sensitive to brane-bending modes $\psi \gg \varphi$.

Numeric solutions of the spherical equations of motion with a point particle source have been obtained elsewhere (4). A major consideration in performing such simulations is that the sources in the save equations are distributional, and hence must be regulated in some way. In (4), the authors regulated the delta-functions by replacing them with thin Gaussians. In Figure 4, we show the results of such a simulation when the perturbing particle is undergoing a periodic orbit. One observes that the GW signal for from the brane is essentially that of a pure massive mode signal.

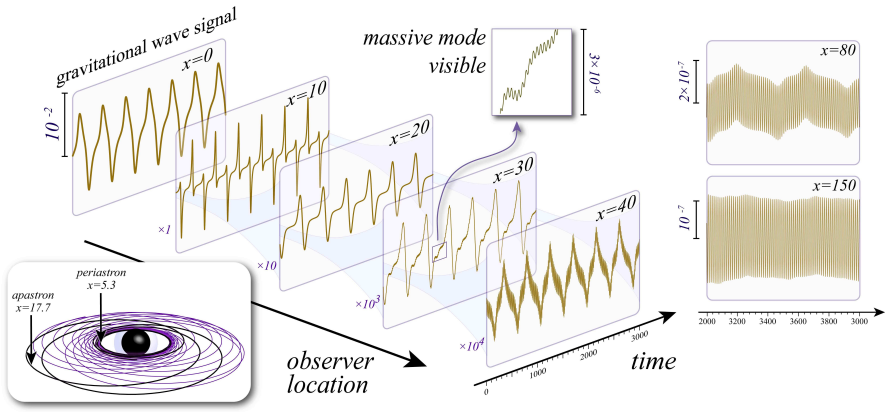

Fig. 4 The steady-state KK gravitational wave signal induced by a particle undergoing a periodic orbit around the black string with $\mu = 0.5$. The orbit (*bottom left*) has eccentricity $e = 0.5$ and angular momentum $p = 3.62$. The waveform of radiation falling into the black string is quite different than that of radiation escaping to infinity: The infalling signal precisely mimics the orbital profile of the source, while the outgoing signal is dominated by monochromatic radiation whose frequency is proportional to the KK mass μ.

10 Estimating the amplitude of the massive mode signal

We have seen in previous sections that if we consider a black string relaxing to its equilibrium configuration or if we look at the GWs emitted by a small particle orbiting the black string, the signal is dominated by massive mode oscillations. The question is: are these oscillations observable? The ability of a GW detector to see a given signal depends on that signal's frequency and its amplitude. The frequency of massive mode signals is well-defined, it is simply given by the solution of the eigenvalue problem presented in §4. However, the amplitude is difficult to pin down unless we consider a specific situation. So in this section, we concentrate on the *s*-wave massive modes emitted by a particle in orbit about a black string. We will be interested in the entire massive mode spectrum; i.e., all values of n. To estimate the GW amplitude associated with heavy modes we will need to analyze the asymptotics of the Green's function solution of the coupled wave equations (118).

10.1 Green's function analysis

The formal solution to the coupled wave equations (118) can be written in terms of the Green's functions

$$(\partial_\tau^2 - \partial_x^2 + V_\psi)G(\tau;x,x') = \delta(\tau)\delta(x-x'), \tag{145a}$$

$$(\partial_\tau^2 - \partial_x^2 + V_\varphi)D(\tau;x,x') = \delta(\tau)\delta(x-x'). \tag{145b}$$

To preserve casuality in the model, we demand that G and D satisfy retarded boundary conditions. That is, they are identically zero if the field point (τ,x) is not contained within the future light cone the source point $(0,x')$.

In terms of these Green's functions, we have

$$\begin{aligned}
\psi(\tau,x) &= \psi_1(\tau,x)+\psi_2(\tau,x),\\
\psi_1(\tau,x) &= \int d\tau' dx'\, G(\tau-\tau';x,x')\mathscr{S}_\psi(\tau',x'),\\
\psi_2(\tau,x) &= \int d\tau' dx'\, G(\tau-\tau';x,x')\hat{\mathscr{F}}(\tau',x')\varphi(\tau',x'),\\
\varphi(\tau,x) &= \int d\tau' dx'\, D(\tau-\tau';x,x')\mathscr{S}_\varphi(\tau',x'). \qquad (146)
\end{aligned}$$

Note the decomposition of ψ into a contribution ψ_1 from the matter source $\mathscr{S}_\psi$, and a contribution ψ_2 from from brane bending φ. These expressions suggest that if we knew the two Green's functions explicitly, the gravitational wave master variable and brane-bending scalar would be given by quadrature.

Unfortunately, G and D are not known in closed form, so we have to resort to numeric computations to accurately calculate the values of ψ and φ induced by a particular source, and for a particular choice of μ. However, any given source will excite all the KK modes to some degree, so to rigourously model the spherical gravitational radiation we would need to do an infinite number of numeric simulations, one for each discrete value of μ. This is not practical, so our goal here is to use the asymptotic behaviour of the propagators to determine the transcendental properties of the emitted radiation and how these scale with the dimensionless Kaluza-Klein mass.

10.2 Asymptotic behaviour

In this subsection, we outline the behaviour of the two retarded Green's functions G and D under the assumption that the the field point is deep within the future light cone of the source point, and is also far away from the string. This is the relevant limit to take if we are interested in the 'late time' gravitational wave signal seen by distant observers.

10.2.1 Brane bending propagator

First, consider the brane bending Green's function. Note that the brane bending potential V_φ is identical to that for the $l=0$ component of a spin-0 field propagating in the Schwarzschild spacetime. This is because the brane bending equation of motion (43) is essentially that of a massless Klein-Gordon field. Fortunately, this propagator has been well studied in the literature, and one can show that

$$D(\tau;x,x') \sim \tau^{-3}, \quad \tau \gg x' - x > 0. \tag{147}$$

This result is most easily interpreted if one considers the initial value problem for φ. That is, we switch off the source in (43) and prepare the field in some initial state on a given hypersurface. Then, a distant observer measuring φ at late times would see the field amplitude decay in time as a power-law with exponent -3.

10.2.2 Gravitational wave propagator

The retarded Green's function for potentials similar to V_ψ have also been considered in the literature. It turns out that the asymptotic character of the potential is the crucial issue. Koyama & Tomimatsu (13) have demonstrated that for potentials of the form

$$V_\psi \xrightarrow[\infty]{r} \mu_n^2 + \mathscr{O}\left(\frac{1}{r}\right), \tag{148}$$

the Green's function has the asymptotic form

$$G(\tau;x,x') \sim \mu_n^{-1/2} \tau^{-5/6} \sin[\mu_n \tau + \phi(\tau)], \quad \tau \gg x' - x > 0. \tag{149}$$

The form of this Green's function rationalizes the waveforms seen in Figure 2, especially the $t^{-5/6}$ envelope of the late time signal, despite the fact that the governing equations (102) were matrix-valued. The key point is the asymptotic form of the potential matrix (106), which says that far from the string the two degrees of freedom are decoupled and governed by a potential of the form (148).

Comparing this expression to the asymptotic form of D above, we see that G decays much slower. This suggests that $\psi_1 \gg \psi_2$ at late times in equation (146); i.e., the portion of the GW signal sourced directly by the stress-energy tensor dominates the brane-bending contribution. Also note the overall $\mu_n^{-1/2}$ scaling of the Green's function with the KK mass of the mode. We will use this below.

10.3 Application to the point particle case for $n \gg 1$: Kaluza-Klein scaling formulae

Let us now use the asymptotic Green's functions in the case where the perturbing matter is a point particle. Our goal is to estimate how the KK signal scales with n for the high mass KK modes.

When the matter stress energy tensor has delta-function support, the $\int d\tau' dx'$ integrals in (146) reduce to line integrals over the portion of the particle's worldline inside of the past light cone. Now, working in the late-time far field limit, we know that the brane-bending contribution to the signal is minimal. We also focus on the high n modes; i.e.,

$$\mu_n \gg 1. \tag{150}$$

Concentrating on the direct signal produced by the particle, we see that the source term $\mathscr{S}_\psi$ for a point particle (9.2) seems to scale as μ_n^{-2}. However, note that the source also involves the derivative of a delta function, which means we must perform an integration by parts. This brings a derivative of G with respect to time into the mix. Again assuming that $\mu_n \gg 1$, we see $\partial_\tau G \sim \mu_n^{1/2} e^{i\mu_n \tau}$. The net result is that we expect

$$\psi \propto \mu_n^{-3/2}, \quad \mu_n \gg 1. \tag{151}$$

That is, all other things being equal, the spherical master variable for a given KK mode scales as $\mu_n^{-3/2}$.

But this is not the entire solution to the problem, since we do not actually observe ψ, we observe h_{AB}. So we need to use the inversion formulae (123) to obtain $h^{(n,s)}_{\alpha\beta}$ and then (87) to get the spherical part of h_{AB}. The detailed analysis leads to the following late-time/distant-observer approximation for the KK metric perturbations:

$$h^{(n,s)}_{AB} \approx h_n \mathscr{F}(t) \sin(\omega_n t + \phi_n)\, \mathrm{diag}\left(0, +1, -\tfrac{1}{2}r^2, -\tfrac{1}{2}r^2 \sin^2\theta, 0\right), \tag{152}$$

where $\mathscr{F}(t)$ is a slowly-varying function of time that depends on the details of the initial data. The characteristic amplitudes h_n are given by

$$h_n = \sqrt{8\pi}\mathscr{A}\left(\frac{2GM_p}{r}\right)\left(\frac{2GM}{\ell}\right)^{-1/2} F_n(d/\ell). \tag{153}$$

Here, r is the distance between the observer and the string, and $\mathscr{A}$ is a dimensionless quantity that depends on the orbit of the perturbing particle but not on n or any other parameters; its value must be determined from simulations. $F_n(d/\ell)$ is a complicated expression involving Bessel functions with the following limiting behaviour: When the perturbing matter is on our brane

$$F_n(d/\ell) \approx \begin{cases} \frac{1}{2} e^{-3d/2\ell} (n\pi^3)^{1/2}, & n \ll 2e^{d/\ell}/\pi^2, \\ e^{-d/2\ell}(n\pi)^{-1/2}, & n \gg 2e^{d/\ell}/\pi^2. \end{cases} \tag{154a}$$

On the other hand, for particles on the shadow brane:

$$F_n(d/\ell) \approx \begin{cases} e^{-d/2\ell}(\pi/2)^{1/2}, & n \ll 2e^{d/\ell}/\pi^2, \\ (n\pi)^{-1/2}, & n \gg 2e^{d/\ell}/\pi^2. \end{cases} \tag{154b}$$

Finally, to a good approximation, the KK frequencies are given by

$$\omega_n = 2\pi f_n \approx \frac{c}{\ell}\left(n + \tfrac{1}{4}\right)\pi e^{-d/\ell}. \tag{155}$$

We note that even though these formulae were derived in the context of the large n approximation, they are actually reasonable approximations to the small n case as well.

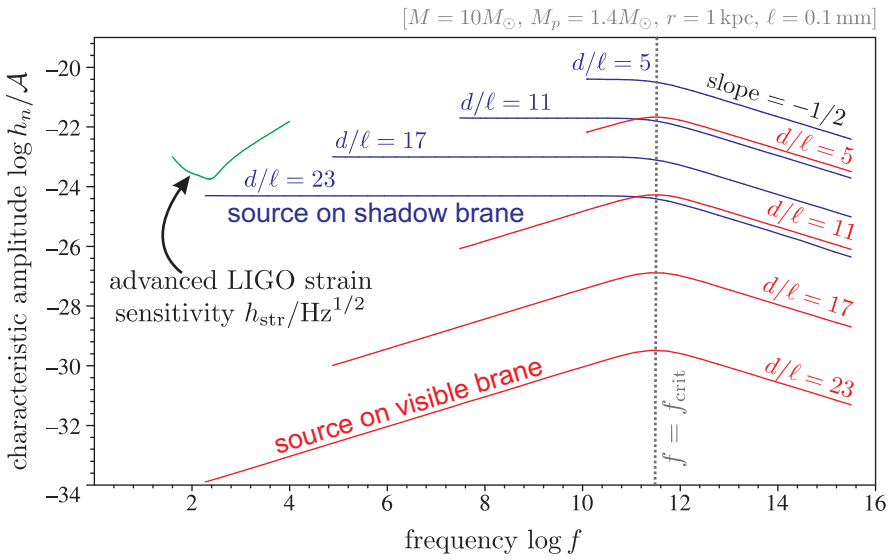

Fig. 5 Characteristic amplitudes of KK radiation emitted by point particles on the visible brane or the shadow brane as follows from equation (153). The particular parameters for this example are indicated just above the plot. Also shown is a dimensionally reduced version of the characteristic strain sensitivity of advanced LIGO for comparison.

10.4 Observability of the massive mode signal

We now have an expression (152) for the amplitude of the spherical massive modes in terms of a parameter $\mathscr{A}$ that can be determined from simulations with μ_n small. This amplitude varies with the type of orbit generating the GWs: it can be $\mathscr{O}(10^{-6})$ or smaller for periodic orbits, or as high as $\mathscr{O}(1)$ for 'zoom-whirl' orbits.[10]

In Figure 5, we plot the characteristic amplitudes h_n as a function of their frequency for a scenario where a $1.4M_\odot$ object is orbiting a $10M_\odot$ black string at a distance of 1 kpc away. Several general trends are obvious:

- the amplitude of the GW signal decreases with increasing brane separation d/ℓ;
- the lowest frequency in the spectrum also decreases with increasing brane separation d/ℓ;
- for a source on the visible brane, the spectrum is peaked about a critical frequency given by

$$f_{\mathrm{crit}} = \frac{1}{\pi^2\ell} \sim 304\,\mathrm{GHz}\left(\frac{\ell}{0.1\,\mathrm{mm}}\right)^{-1}; \tag{156}$$

- when the perturbing particle is on the shadow brane, the spectrum is flat underneath the critical frequency f_{crit}; and,

[10] These are orbits where the particle comes in from infinity, is briefly captured by the black string, and then escapes to infinity again.

- in all cases, the signal from shadow particles is stronger than that of visible particles.

In general, the peak amplitude $h_{\max}$ is the one corresponding to the critical frequency, and is given by

$$h_n \leq h_{\max} \sim \mathscr{A}\left(\frac{M_p}{M_\odot}\right)\left(\frac{r}{\mathrm{kpc}}\right)^{-1}\left(\frac{M}{M_\odot}\right)^{-1/2}\left(\frac{\ell}{0.1\,\mathrm{mm}}\right)^{1/2} \times \begin{cases} 5.0\times 10^{-22} e^{-(d-5\ell)/\ell}, & \text{visible source,} \\ 9.1\times 10^{-21} e^{-(d-5\ell)/2\ell}, & \text{shadow source.} \end{cases} \tag{157}$$

Figure 5 illustrates the main problem with observing the KK signal from a black string. The frequencies in the KK spectrum are bounded below by

$$f_n \geq f_{\min} \sim 12\,\mathrm{GHz}\left(\frac{\ell}{0.1\,\mathrm{mm}}\right)^{-1} e^{-(d-5\ell)/\ell}. \tag{158}$$

This implies that the KK spectrum is usually in a higher waveband that the operation frequencies of LIGO and LISA, assuming that $\ell \lesssim 50\,\mu$m in line with current experimental tests. The way to mitigate this is to push the branes farther apart, which reduces $f_{\min}$. But if one does this, the amplitude of the signal goes down exponentially. Clearly, the situation is much better for shadow particles, which have an intrinsically stronger GW signal. The detailed prospects of observing massive mode signal with realistic GW detectors is discussed in (4).

11 Summary and outlook

In these lecture notes, we have introduced the black string braneworld, which is a candidate model for a brane black hole in the Randall-Sundrum scenario. At the background level, this model is indistinguishable from the Schwarzschild solution to brane observers, so we need to examine the perturbations of the model to find deviations from general relativity. We have developed the formalism necessary to calculate the gravitational wave signals emitted from black strings perturbed away from their equilibrium configurations. We have found that the late time nature of these signals is somewhat independent of the nature of the mechanism which generated them, and is a long-lived superposition of discrete monochromatic massive modes. We have discussed how these massive modes could be produced by a point particle orbiting a black string, and estimated what their amplitude might be.

There are a number of open issues that need to be addressed in this model. So far, we have only been able to estimate amplitudes by analyzing the scaling behaviour of Green's functions and using point particle sources. We need to confirm our scaling results with direct simulations and we need to move beyond the point particle approximation to model realistic sources with size larger than ℓ. The phenomenon of

localized black hole-black string transitions must be looked at in quantitative detail. The possibility that such a phase transition can produce significant amounts of massive mode radiation and contribute to the gravitational wave background provides one of the best prospects for the actual detection of a black string.

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