MATH 4503: Convergence & Stability of ODE stencils

Consider an ODE $y' = f(x, y(x))$. A generic stencil to solve this ODE is

$$y_{i+1} = y_i + h\Psi(h, x_i, y_{i+1}, y_i, \cdots, y_{i-k}) \quad (1)$$

Here, $y_i \approx y(x_i)$ is the numeric approximation to the true sol$^{\underline{n}}$ $y(x)$ at $x = x_i = x_0 + ih$. For example, in the forward Euler

$$y_{i+1} = y_i + hf(x, y_i), \quad \Psi = f$$

If the true sol$^{\underline{n}}$ $y = y(x)$ is subbed into (1), there is no strict equality between the LHS & RHS:

$$y(x_{i+1}) = y(x_i) + h\Psi\left(h, x_i, y(x_{i+1}), \cdots, y(x_{i-k})\right) + \tau_i h^{p+1} \quad (2)$$

Here, $\tau_i h^{p+1}$ is the "one-step error" in the numeric stencil. The "local truncation error" is the one step error divided by h. For the Euler method,

$$\tau_i h^{p+1} = \frac{1}{2} y''(x_i) h^2 + \mathcal{O}(h^3)$$

We define $\tau_i = \mathcal{O}(1)$, so here we see $p = 1$ & $\tau_i = \frac{1}{2} y''(x_i) + \mathcal{O}(h)$. The global error is defined as

$$E_n = y_n - y(x_n) \quad (3)$$

Which yields

$$E_{i+1} = E_i + h\left[\Psi(h, x_i, y_{i+1}, \cdots) - \Psi(h, x_i, y(x_{i+1}), \cdots)\right] - \tau_i h^{p+1} \quad (4)$$

We say a method is consistent if

$$\lim_{h\to 0} \frac{y_{i+1}-y_i}{h} = f(x_i, y_i)$$

We say a method is convergent if

$$\lim_{h\to 0} E_n = 0$$

We say a method is zero-stable if

$$E_n = \mathcal{O}(h^p) \text{ as } h\to 0$$

Let's now specialize to explicit one-step methods (like Runge-Kutta)

$$(5)\quad y_{i+1} = y_i + h\Psi(h, x_i, y_i)$$

$$(6)\quad E_{i+1} = E_i + h\left[\Psi(h, x_i, y_i) - \Psi(h, x_i, y(x_i))\right] - \tau_i h^{p+1}$$

In order to place a bound on E_i, we need to assume the Ψ is Lipschitz continuous:

$$|\Psi(h, x_i; y_1) - \Psi(h, x_i; y_2)| \leq L|y_1 - y_2|, \quad \forall y_1, y_2 \in \mathbb{R}$$

→ kind of like a bound on the first derivative of Ψ when y_1 close to y_2, but a little stronger

For example, if we use Forward Euler to solve $y' = \lambda y$,

$$\Psi = \lambda y \Rightarrow |\Psi(y_1) - \Psi(y_2)| = |\lambda||y_1 - y_2|$$

$$\Rightarrow L = |\lambda|$$

Assume Lipshitz cond^n, the triangle inequality gives for (6)

$$|E_{i+1}| \leq |E_i| + hL|y_i - y(x_i)| + |\tau_i|h^{p+1}$$

$$= |E_i|(1+hL) + |\tau_i|h^{p+1} \qquad (7)$$

The formula (7) can be applied recursively

$$
\begin{aligned}
|E_n| &\leq |E_{n-1}|(1+hL) + |\tau_{n-1}|h^{p+1} \\
&\leq \left[|E_{n-2}|(1+hL) + |\tau_{n-2}|h^{p+1}\right](1+hL) + |\tau_{n-1}|h^{p+1} \\
&\leq \cdots \\
&\leq |E_0|(1+hL)^n + h^{p+1}\sum_{k=0}^{n-1}(1+hL)^{n-k-1}|\tau_k|
\end{aligned}
$$

Now, we note that $(1+hL) \leq e^{hL}$ and assume $|\tau_k| \leq \tau_\infty$

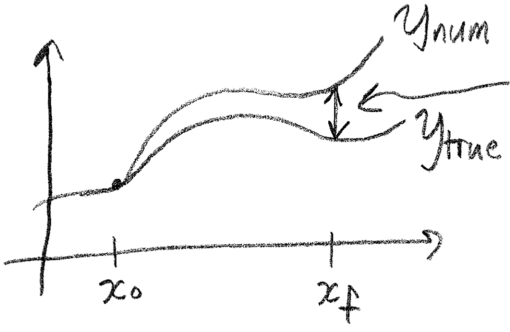
$$|E_n| \leq |E_0|(1+hL)^n + h^{p+1}\sum_{k=0}^{n-1} e^{hL(n-k-1)}\tau_\infty$$

But $e^{hL(n-k-1)} \leq e^{nhL}$ $\Rightarrow$ $|E_n| \leq |E_0|(1+hL)^n + h^{p+1}\tau_\infty e^{nhL} n$

Note that $E_0 = y_0 - y(x_0) = 0$ since we impose initial data directly.

Also note $\dfrac{x_n - x_0}{h} = n \Rightarrow |E_n| \leq h^p \tau_\infty (x_n - x_0) e^{L(x_n - x_0)}$

Hence, we have just shown that all explicit-one step methods are zero stable if Ψ is Lipshitz cts. $\square$



$|E(x_f)| \leq h^p \tau_\infty (x_f - x_0) e^{L(x_f - x_0)}$

This says that the error between the true and numeric sol^n at a <u>given</u> value of x_f is $\mathcal{O}(h^p)$ as $h \to 0$