ADAPTIVE STEPSIZE INTEGRATORS

Suppose we have 2 numeric stencils (explicit & one-step) to solve

$$y' = f(x,y)$$

$$y_{n+1}^{(1)} = y_n + h\Psi_1(h, x_n, y_n)$$
$$y_{n+1}^{(2)} = y_n + h\Psi_2(h, x_n, y_n)$$

Method 2 is more accurate than method (1):

$$y(x_{n+1}) = y(x_n) + h\,\Psi_1(h, x_n, y(x_n)) + \tau_n^{(1)} h^p$$
$$y(x_{n+1}) = y(x_n) + h\,\Psi_2(h, x_n, y(x_n)) + \tau_n^{(2)} h^{p+k}$$

Let us regard $y_n = y(x_n)$; i.e. let's neglect the error in y_n. Then,

$$y(x_{n+1}) = y_n + h\Psi_1(h, x_n, y_n) + \tau_n^{(1)} h^p$$
$$y(x_{n+1}) = y_n + h\Psi_2(h, x_n, y_n) + \tau_n^{(2)} h^{p+k}$$

or

$$y(x_{n+1}) = y_{n+1}^{(1)} + \tau_n^{(1)} h^p$$
$$y(x_{n+1}) = y_{n+1}^{(2)} + \tau_n^{(2)} h^{p+k}$$

$$\Rightarrow \quad y_{n+1}^{(2)} - y_{n+1}^{(1)} = \tau_n^{(1)} h^p - \tau_n^{(2)} h^{p+k} = \tau_n^{(1)} h^p\left[1 + \mathcal{O}(h^k)\right]$$

Neglecting the higher order terms, we see that

$$y_{n+1}^{(2)} - y_{n+1}^{(1)} \approx \tau_n^{(1)} h^p = \text{local error in stencil } \#1 = \varepsilon_n(h)$$

Now, we know the local error is prop to h^p, so if we were to use a new stepsize

$$\varepsilon_n(h^*) = \frac{h_*^p}{h^p}\,\varepsilon_n(h)$$

Let's now choose $\varepsilon_n(h^*) = \varepsilon_* =$ a user defined tolerance. Then

$$h_* = \left[\frac{\varepsilon_*}{y_{n+1}^{(2)} - y_{n+1}^{(1)}}\right]^{1/p}$$

Hence, if we use stencil 2 again with this new stepsize, we are guaranteed that the local error will be less than our selected tolerance.